Measuring Freedom in Games

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Abstract

Behind the veil of ignorance, a policy maker ranks combinations of game forms and information about how players interact within the game forms. The paper presents axioms on the preferences of the policy maker that are necessary and sufficient for the policy maker's preferences to be represented by the sum of an expected valuation and a freedom measure. The freedom measure is the mutual information between players' strategies and the players' outcomes of the game, capturing the degree to which players control their outcomes. The measure extends several measures from the opportunity set based freedom literature to situations where agents interact. This allows freedom to be measured in general economic models and thus derive policy recommendations based on the freedom instead of the welfare of agents. To illustrate the measure and axioms, an application to civil liberties is provided.

Keywords: Freedom of Choice, Mutual Information, Entropy, Measurement, Game Form, Process, Civil Liberties

JEL Classification: D63, D71, D81

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1 Introduction

This paper axiomatically analyzes a decision problem of institutional choice in which a policy maker only has information about how individuals will interact within a game form but does not know unobservable information. In such settings, utilitarianism is infeasible as it relies on knowledge of utilities which are not observable. This setting is plausible for institutions that last far into the future and thus utilities are hard to estimate, institutions that have nonrational agents without well-defined utility functions, and institutions in which the procedure by which an outcome is implemented is intrinsically valuable.

We call a game form endowed with the policy maker's information about how players will interact a *process*. The information about players' strategies allows us to establish how well their strategies correlate with outcomes and how quantitatively diverse the choices are. The game form allows us to establish that these correlations are indeed causal. Together, the two pieces of information allow the decision maker to account for the control players have over the outcomes.

The policy maker forms preferences over processes in compliance with the following axioms. The Rationality axiom imposes completeness and transitivity of the preference relation. Continuity and Outcome Equivalence ensure that similar processes are similarly ranked. Lottery Independence requires the policy maker to obey the von Neumann-Morgenstern independence axiom for pure lotteries over outcomes.

The central axioms are Strategy Independence and Subprocess Monotonicity. Strategy Independence deals with situations in which the policy maker learns that a choice between strategies was actually made by nature. Thus, instead of a player making a choice between strategies, nature randomly chooses the strategy for the player. The axiom requires that ceteris paribus, the change of value due to this choice removal is independent of the other choices being made by any player. For example, if the policy maker learns that aversion to bitter vegetables is determined genetically (Wooding et al., 2004), then the resulting change in the policy maker's preference is independent of the policy maker's preference change resulting from learning that smoking behavior is genetically determined (Erzurumluoglu et al., 2019). Precisely, the policy maker may not prefer that vegetable choices are determined by nature if and only if smoking choices are also determined by nature. Instead, the policy maker needs to make independent judgments about the desirability of the agent (rather than nature) being in control of strategic choices.

Subprocess Monotonicity requires the policy maker's preference for a process to be increasing in the preference of its subprocesses. A subprocess is the process obtained from conditioning the probability measures on behavior to a subgame. Monotonicity in the value of a subprocess is only required to hold when the subprocess reaches distinct outcomes from the remainder of the game. Consider as a simple example the process in which a single player gets to choose between smoking and not smoking. According to the information of the policy maker, both smoking and not smoking are equally likely to be chosen. This process has two trivial subprocesses, one in which the player smokes with certainty and one in which the player does not smoke with certainty. Suppose the policy maker prefers to dictatorially assign not smoking to dictatorially assigning the player to smoke. Then subprocess monotonicity without the requirement of disjoint outcomes would imply that the substitution of the subprocess in which the player smokes with certainty by a subprocess in which the player does not smoke with certainty would improve the process. However, the resulting process would be the trivial choice between not smoking and not smoking. When we substitute a subprocess by another subprocess it may therefore occur that meaningful choices are removed if some of the outcomes of the subprocess overlap with outcomes of the remainder of the process. Therefore, we require Subprocess Monotonicity only to hold if the outcomes of the subprocess are disjoint from the remainder of the game.

We obtain a representation theorem according to which the policy maker's preferences are additively separable across players. For each player, the policy maker's evaluation of the process consists of the sum of two components. The first component is an expectation across the valuation of individual outcomes that can be interpreted as the policy maker's perceived instrumental value of the process.[1] The second component is the mutual information between the player's strategies and outcomes. This component is interpreted as a freedom measure; it measures the degree to which players exercise control over their outcomes. Under the special case of perfect control, the mutual information becomes equal to the Shannon entropy of the outcomes, a freedom of choice measure suggested by Suppes (1996). The policy criterion has several desirable properties. First, by allowing for interactions between players, the policy maker

[1]Since the policy maker has no information about player's utilities beyond the behavior, this expectation is the policy maker's subjective evaluation of how desirable the outcomes are, not the player's.

can have procedural preferences, e.g., preferences over who can influence what outcome. Second, the criterion only depends on behavioral data and does not depend on unobservable quantities such as utility. Third, no impositions are made with respect to equilibrium concepts or the rationality of players in the game. Indeed, the policy maker need not even know whether a decision maker is rational.

The contribution to the freedom of choice literature[2] is a solution to the problem posed in Pattanaik (1994). Pattanaik (1994) showed that opportunity set based measures of freedom of choice encounter problems when being applied to situations in which agents interact. The difficulty arises because in situations in which agents interact, opportunity sets from which agents can freely choose are no longer clearly defined. The choice of one agent may influence the available opportunities of another agent and vice versa. This problem has prevented the literature to provide measures even for a simple exchange economy as Pattanaik (1994) showed. Yet, it is exactly these cases when agents depend on each other to achieve their goals, when they exhibit power over each other, or when they are coerced by others that the measurement of freedom becomes interesting. The lack of freedom measures for situations where agents interact therefore creates an undesirable wedge between the normative analysis that can be performed by economists and normative perceptions outside economics.

To show that the measure axiomatized in this paper effectively solves the problem of measuring freedom when agents interact, we apply the measure to a simplified model of racial discrimination on buses in Montgomery in the early 1950s. We analyze a game form representing the interaction between a passenger and a driver. Using historical accounts, we can inform a policy maker about how the passengers and drivers interacted. According to the law, no passenger had to yield their seat to another passenger. However, black passengers were frequently required to yield their seats for white passengers. In case they refused to yield their seat, they were arrested and economically sanctioned. We show how this discrimination leads to a reduction of freedom of choice. The example also shows why we include the policy maker's information about the strategies of the players; neither a game form in which only the *legal* actions are included, nor a game form in which all *possible* actions are included would correctly capture the degree of freedom of choice of the players.

The closest in spirit to our model is the literature on freedom of choice in

[2]Dowding and van Hees (2009) gives a survey of the literature. Appendix 4 presents freedom of choice measures that relate to the measure developed here.

game forms (Ahlert, 2010; Bervoets, 2007; Braham, 2006; Peleg, 1997). Moving from opportunity sets to a more general framework was an important conceptual innovation. This moved the quest for a proper measure of freedom from measuring numbers of alternatives to measuring control over choice. To this end, Gustafsson (2010), Sher (2018) evaluate the qualitative diversity offered by sets of lotteries. Evaluating control is important since cases of actual policy relevance (discrimination, consumer freedom, political participation) are unlikely purely decision theoretic; difficult policy tradeoffs commonly involve the freedoms of multiple individuals. Suppes (1996) suggests using a measure of quantitative diversity (Nehring & Puppe, 2009) to measure freedom of which the measure in the paper is an extension to a domain that allows for imperfect control over outcomes.

The paper continues as follows. The Montgomery bus game as an example of civil liberties is introduced in Section 2. Section 3 begins with an informal description of the policy maker's problem and then provides the game theoretic framework in which the measure is developed. Section 4 axiomatizes the measure.

2 Example: Civil Liberties

To clarify the various concepts used and defined in the following sections, we employ an example of discrimination. The extensive game form[3] is shown in Figure 1. In the game player 1 decides how to get to work. She can choose to walk, in which case she arrives with certainty at work, outcome z. Alternatively, she can attempt to take the bus. In that case, the driver, player 2, decides whether to drive past her or stop and accept her as a passenger. If player 1 is rejected as a passenger, she has the choice of walking and arriving with a delay, x, or cancel the trip, y. If player 1 does not get passed by the driver, she is at one point during the ride requested to give up her seat to a white passenger. She can then yield her seat to the other passenger and stand for the rest of the ride, u. Alternatively, she can insist on her right to sit, in which case player 2 can either act lawfully and player 1 gets to sit during the ride, v, or player 2 can call the police and in the aftermath player 1 gets arrested and loses her job, w.

[3] A similar game form is given in Mailath et al. (1993). The interpretation added in this paper to the game form is a much simplified account of the segregation laws and discriminatory practices of bus lines in Montgomery, Alabama up to 1956. For historical accounts, see Burns (2012), Phibbs (2009), Theoharis (2015).

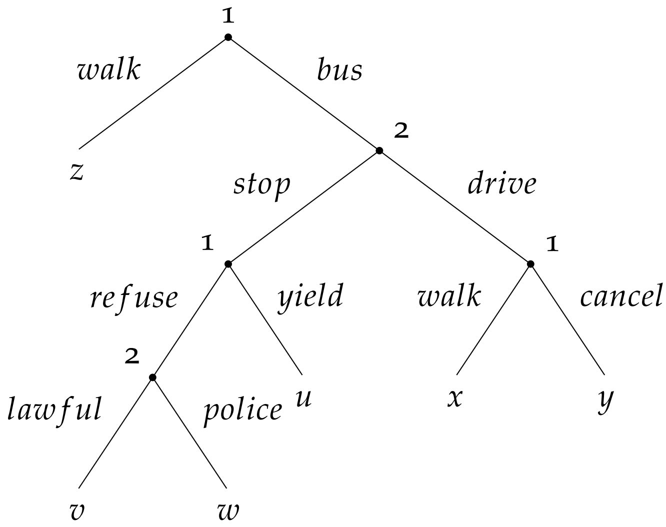

Figure 1: Montgomery Bus Extensive Game Form

A particular instance of the Montgomery bus game was played on December 1, 1955 between Rosa Parks and and James Blake, famously ending in outcome w, which led ultimately to the Montgomery bus boycott and a change in the segregation rules. From an opportunity based freedom of choice perspective in addition to her chosen action "refuse", Rosa Parks clearly also had the freedom to choose "yield". This stands in contrast to our intuition that in general passengers did not have much freedom when they interacted in this institution. The reason for this is that only a handful of courageous women[4] dared to take the action "refuse" and suffered hardship as a consequence of taking this action. Based on historical accounts of 1950s Montgomery, Alabama, we know that bus drivers at times arbitrarily rejected black passengers and black passengers would frequently be required to yield their seat for white passengers. White passengers on the other hand received preferential treatment by the bus drivers. Therefore, the frequency with which actions are taken by various individuals in the same role within an institution, can be informative about the freedom an institution offers.

We do not directly address the usual question in the freedom of choice literature of how much freedom a particular individual has in a particular choice situation. Instead, we measure how much freedom of choice an institution offers to the different roles individuals take within that institution. Thus, our measure can give an answer to how much freedom of choice passengers and

[4]Claudette Colvin, Aurelia Browder, Susie McDonald, Mary Louise Smith, Jeanetta Reese, and Rosa Parks.

bus drivers had in 1950's Montgomery but this may only be indicative of the freedom of choice Rosa Parks and James Blake had in the specific instance when they interacted in this institutional setting. Therefore, a player in the game form is best understood as a role within the institution and not as a particular individual.

One repugnant aspect about the institution represented by the Montgomery bus game is that whether player 1 ends up in outcome v, sitting on the bus, or outcome w, sitting in prison, causally depends on player 1's race and not their own strategies. We can model this via an additional move by nature determining the race of the player. If the policy maker dislikes discrimination, then uncertainty about strategies and uncertainty about moves by nature must be valued differently. The policy maker therefore has source-dependent preferences (Chew & Sagi, 2008) about uncertainty with respect to strategic uncertainty and risk generated by nature. To ensure that causal relations between strategies and outcomes are unambiguous, we require strategies to be uncorrelated across all players, including nature. The structure of the game form ensures that the measured correlation between strategies and outcomes indeed reflects the control a player has over the outcomes.

Thus, we can measure the degree to which players' strategies cause outcomes to occur by the degree to which the conditional probability of an outcome changes in response to a change in strategy. If player 2 with some probability chooses the pure strategy "police" and with some probability the pure strategy "lawful" irrespective of nature's move determining the race, then the conditional probability of outcomes v and w given strategies will be very different from their unconditional probability. This can more generally be measured using the mutual information between strategies and outcomes.

3 The Model

A policy maker faces a decision problem in which she decides behind a veil of ignorance between establishing different institutions. The institutions are modeled as game forms between a set of players over lotteries of *social outcomes*. Players in the game form are understood as representing roles within an institution and behind the veil of ignorance their identity is uncertain. The policy maker has information about how players will interact in the game form. More precisely, the policy maker is given a probability measure over the (possibly mixed) strategies of each player. The combination of a game form with the policy maker's information about strategies is called a *process*. Processes can differ on the game form, contain identical game forms but different information about strategies, or differ in both respects. In this section we will make this definition mathematically precise.

3.1 Notation

f, g, h denote generic functions $f : x \mapsto f[x]$. $f|_{\mathcal{Z}}$ denotes the restriction of $f : \mathcal{X} \to \mathcal{Y}$ to the subset $\mathcal{Z} \subseteq \mathcal{X}$ of the domain. If $f : \mathcal{X} \to \mathcal{Y}$, then $f[\mathcal{Z}] = \{y \in \mathcal{Y} : \exists z \in \mathcal{Z} : y = f[z]\}$ is the image of the function of the set $\mathcal{Z} \subseteq \mathcal{X}$. When a set $\mathcal{X}$ is understood as a subset of $\mathcal{Y}$, then $\mathcal{X}^C = \mathcal{Y} \backslash \mathcal{X}$ denotes the complement.

If $\mathcal{S}$ is a topological space, then $\Delta\mathcal{S}$ denotes the finite support probability measures over the Borel sigma algebra of $\mathcal{S}$. The support of $\nu \in \Delta\mathcal{S}$ is denoted by $supp[\nu]$. In case of a finite set $\mathcal{S}$, we assume the discrete topology and therefore $\nu \in \Delta\mathcal{S}$ means the domain of ν is the power set $2^{\mathcal{S}}$. We will frequently simplify notation by writing $\nu[s]$ instead of $\nu[\{s\}]$ for singletons. If $s \in \mathcal{S}$, then $\mathbb{1}_s \in \Delta\mathcal{S}$ fulfills $\mathbb{1}_s[s] = 1$. If $\nu \in \Delta\mathcal{S}$, $\mathcal{S}' \subset \mathcal{S}$, and $\nu[\mathcal{S}'] > 0$, then the conditional probability measure denoted by $\nu|\mathcal{S}'$ fulfills $(\nu|\mathcal{S}')\,[\mathcal{S}''] \cdot \nu[\mathcal{S}'] = \nu[\mathcal{S}'']$ for $\mathcal{S}'' \subseteq \mathcal{S}'$.

For any two probability measures $\nu \in \Delta\mathcal{S}, \nu' \in \Delta\mathcal{S}'$, we can assign a product measure $\nu \otimes \nu' \in \Delta(\mathcal{S} \times \mathcal{S}')$, such that $(\nu \otimes \nu')[s, s'] = \nu[s]\nu'[s']$. For finitely many products of a set of measures, $\mathcal{D} = \{\nu_1, ..., \nu_n\}$, we can write $\bigotimes_{\nu \in \mathcal{D}} \nu = \nu_1 \otimes ... \otimes \nu_n$.

For any two probability measures over the same set $\nu, \nu' \in \Delta\mathcal{S}$, we can define the mixture of the two probability measures $\alpha\nu \oplus (1-\alpha)\nu' \in \Delta\mathcal{S}$ as the probability measure that fulfills for all $s \in \mathcal{S}$: $\alpha\nu \oplus (1-\alpha)\nu'[s] = \alpha\nu[s] + (1-\alpha)\nu'[s]$. For a probability measure $\alpha \in \Delta\mathcal{S}'$ and an injective function $f : \mathcal{S}' \to \Delta\mathcal{S}$, we define

$$\begin{aligned}\bigoplus_{s'} \alpha[s']f[s'] =& \alpha[s'_1]f[s'_1] \oplus (1-\alpha[s'_1]) \left(\frac{\alpha[s'_2]}{1-\alpha[s'_1]} f[s'_2] \oplus \dots \right) \\ =& \alpha[s'_1]f[s'_1] \oplus \alpha[s'_2]f[s'_2] \oplus \alpha[s'_3]f[s'_3] \oplus \dots \end{aligned} \tag{1}$$

In other words, f and α can together be interpreted as a two stage probability measure over $\mathcal{S}$ and the measure $\bigoplus_{s'} \alpha[s']f[s']$ is its reduction to a single stage.

3.2 Game Forms and Processes

Let $\mathcal{N}$ be a set of players. We assume there exists some universal set of social outcomes $\mathcal{O}$. Outcomes are denoted by lowercase letters from the end of the alphabet, x, y, z. For each player i, there exists a set $\mathcal{O}_i$ of individual outcomes $x_i, \ldots$ that are a partition of $\mathcal{O}$. If the policy maker is of the opinion that the difference in outcomes x and y are irrelevant[5] for player i, then the individual outcome of player i is the same in both outcomes, i.e., $\exists x_i \in \mathcal{O}_i : x, y \in x_i$. For simplicity, we assume that all combinations of individual outcomes are possible, i.e., for all $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} \mathcal{O}_i$, we have that $\bigcap_{i \in \mathcal{N}} x_i \neq \emptyset$.

We define strategic game forms as follows.

Definition 1 (Strategic Game Form)**.** A strategic game form G is a tuple $(\mathcal{N}, \mathcal{A}, o)$ where

— $\mathcal{N} = \{1, ..., n\}$ is a finite set of players.

— $\mathcal{A} = \prod_{i \in \mathcal{N}} A_i$ is the set of action profiles.

— $\mathcal{A}_i$ is the finite set of actions of player i.

— $o : \mathcal{A} \to \Delta\mathcal{O}$ is the outcome function specifying for each action profile a lottery.

The set of all strategic games in reduced form, i.e., without any strategically equivalent actions[6] of any player is denoted by $\mathcal{G}[\mathcal{O}]$.

Lowercase letters from the beginning of the alphabet $a, b, c, \ldots \in \mathcal{A}$ always denote action profiles, action profiles with a subsript, $a_i \in A_i$ denote the action taken by player i. To avoid the awkward notation $(a_1, ..., a_{i-1}, a_i, a_{i+1}, ..., a_n)$ where $a_j \in \mathcal{A}_j$, we employ the notation (a_i, a_{-i}) for such tuples. Each action profile results in a lottery over outcomes. In the following, G always denotes an arbitrary reduced game form with the same set of players $\mathcal{N}$.

Example. The reduced normal game form of the Montgomery bus extensive game form is shown in Table 1. Each action is now spelled out as a fully contingent plan. We note that in this game there is no uncertainty about the outcomes after the players' moves. Therefore, for example $o[a_1^1, a_2^1] = \mathbb{1}_u$. We

[5] Ahlert (2010) instead employs a perception function that distinguishes social states according to whether individuals perceive the states to be different. Instead of making this a question of perception, we make this a normative issue to be determined by the policy maker.

[6] For a precise definition, see Appendix **??**.

		stop, lawful	stop, police	drive
		a_2^1	a_2^2	a_2^3
bus, yield, walk	a_1^1	u	u	x
bus, yield, cancel	a_1^2	u	u	y
bus, refuse, walk	a_1^3	v	w	x
bus, refuse, cancel	a_1^4	v	w	y
walk	a_1^5	z	z	z

Table 1: Montgomery Bus Game Form

must also determine what the relevant distinctions in outcomes are. Based on the above interpretation, we may for example be concerned about the freedom of choice of player 1 about whether and how she commutes to work. This means that u, v, etc., will generally be considered normatively distinct outcomes for player 1 by the policy maker and therefore belong to different elements of the partition $\mathcal{O}_1$. Some policy maker perhaps considers the delay from being rejected from riding the bus negligible and considers the outcomes z and x equivalent for player 1. Another policy maker may find that the driver's outcomes are all identical $\mathcal{O}_2 = \{\{u, v, w, x, y, z\}\}$. In any case, these normatively imposed distinctions are assumed to be exogeneously given.[7] *End of example.*

The information contained in a game form is not always sufficient to make moral judgments. For example, a utilitarian policy maker would in addition like to know the utilities and the expected behavior of individuals. We will not permit utility information behind the veil of ignorance but information about player's behavior only. We define the mixed strategies μ_i of player i as probability measure over the actions $\mu_i \in \Delta\mathcal{A}_i$ and strategy profiles as $\mu = (\mu_i)_{i\in\mathcal{N}}$. With some abuse of notation, we will also use μ as the corresponding product measure over action profiles.

Behind the veil of ignorance, the policy maker is uncertain about the choice of strategies of the player. This uncertainty is reflected in the behavior of the player. We therefore define the information of the policy maker as probabilistic beliefs.

Definition 2 (Information about Strategies)**.** The information of the policy maker about the strategies of player i, $\theta_i \in \Delta\Delta\mathcal{A}_i$ is a finite support probability measure over the strategies, $\Delta\mathcal{A}_i$. The policy maker's information about strategy

[7] As already argued by Sugden (2003), any measure of freedom of choice ultimately depends on the way the outcome space is partitioned. We will see in Section **??** how different partitions of the outcome space yield different policy objectives.

profiles is given by $\theta = (\bigotimes_{i \in \mathcal{N}} \theta_i) \in \prod_{i \in \mathcal{N}} \Delta\Delta\mathcal{A}_i$.

It is noteworthy that the information about strategies exhibits independence across players. This implies that the description of the interaction provided by the game form is *comprehensive* in the following sense. According to the information of the policy maker, the players have no way of correlating their actions in any way that is not described by the game form. As we are interested in process value, this is important as the following example shows.

Example. Suppose in the Montgomery bus game form we observe that according to the information of the policy maker, player 1 plays the pure strategy a_1^5 if and only if player 2 plays a_2^2 and player 1 plays a_1^3 if and only if player 2 plays a_2^1. In this case, it seems that player 1 has perfect control over whether she walks, z, or sits on the bus, v. Similarly, player 2 seems to have perfect control over these two outcomes as well. This cannot be the whole story of the interaction between the two players, however. To coordinate their actions, they must either rely on a signal from nature (let's say weather) or one player is able to condition on the other's strategy. The policy maker may want to judge the two situations differently. However, only knowing that the actions are correlated but not the reason for the correlation makes it impossible to determine which of the two scenarios is correct.

Consider the first scenario: on days with hot weather, player 1 walks and the driver is in a foul mood and would discriminate if player 1 tries to enter the bus. On days with cool weather, player 1 takes the bus and player 2 is in a good mood and does not discriminate. This scenario would be properly modeled by a process mixture (introduced below) in which with some probability the players interact in a process modeling the good weather interaction and with the remaining probability the bad weather interaction. In this process mixture, which is itself a process, the strategies would be uncorrelated.[8]

Alternatively, consider a second scenario: player 1 observes whether today's driver is known to discriminate and avoids taking the bus. If player 2 observes that today's driver is known not to discriminate, she takes the bus. This process would be properly modeled by accounting for the fact that player 2 indeed plays a pure strategy that is a contingent plan depending on the identity of

[8]Similarly, if the decision maker is uncertain about which equilibrium will be played in a game, this is properly addressed by a process mixture across several processes, one for each equilibrium. Also, correlated equilibria would need to be modeled by making the correlation device explicit via process mixtures.

the driver. In a process that accounts for this, the strategies would again be uncorrelated.

We therefore impose that the probability measure over strategy profiles is a product measure of the probability measures over individual strategies. By assuming this, we can identify the causal relations between each player's strategies and the outcomes since all correlations must be either due to moves by nature or from the modeled strategic interactions. *End of example.*

We now define the primitives over which the policy maker has preferences. These primitives determine what the policy maker sees behind the veil of ignorance. The policy maker forms preferences over processes from a set of processes defined as follows.

Definition 3 (Set of Processes)**.** The set of processes $\mathcal{P}$ is defined as

$$\mathcal{P} = \{(G, \theta) : G = (\mathcal{N}, \mathcal{A}, o) \in \mathcal{G}[\mathcal{O}], \theta \in \prod_{i=1}^{n} \Delta\Delta\mathcal{A}_i\}. \tag{2}$$

Thus, processes are game forms endowed with information about strategies and the set $\mathcal{P}$ contains all possible processes given the set of outcomes $\mathcal{O}$.

Example. In the Montgomery Bus Game, it is quite plausible that a policy maker's evaluation will generally depend on how the individuals interact. If a policy maker expects the driver to always reject the passenger and the passenger always chooses to walk, then the policy maker may conclude that freedom of choice for the passenger is lower than if the policy maker expects the driver to stop for the passenger. Thus, the behavior of player 2 is important for evaluating the freedom of choice for player 1. However, the behavior of player 1 is similarly important. Suppose the driver sometimes stops and sometimes drives past. If the threat of the driver rejecting the passenger is sufficiently severe such that she always chooses to walk. Then we may also judge that freedom of choice is low – this time not from the expectations about the driver's behavior but from the behavior of the passenger. *End of example.*

In the following, we define useful concepts related to processes. We introduce the following notation for the probability measure over outcomes derived from a process (G, θ):

$$\rho_{G,\theta} = \bigoplus_{\mu} \theta[\mu] \bigoplus_{a} \mu[a] o[a] \tag{3}$$

$$\rho_{G,\theta}|\mu_i = \bigoplus_{\mu_{-i}} \theta_{-i}[\mu_{-i}] \bigoplus_{a_i \in \mathcal{A}_i, a_{-i} \in \mathcal{A}_{-i}} \mu_i[a_i]\mu_{-i}[a_{-i}] o[a_i, a_{-i}] \tag{4}$$

which are well-defined measures defined via mixtures of measures. The definition of $\rho_{G,\theta}$ mixes first all probability measures of outcomes given action profiles, $o[a]$, with weight $\mu[a]$ into a conditional probability of outcomes given a profile μ. It then mixes all the thereby resulting measures given μ with the weight $\theta[\mu]$. For an outcome x, $\rho_{G,\theta}[x]$ thus denotes the probability of how likely it is that outcome x occurs in a game form G with information θ about the strategies played by players. The definition of $\rho_{G,\theta}|\mu_i$ proceeds in the same manner but fixes player i's strategy, μ_i.

Not all outcomes of a game form are reached with positive probability. In a process (G, θ), the support of an action profile $a \in \mathcal{A}$ from the perspective of $i \in \mathcal{N}$ is defined as:

$$supp_{i,(G,\theta)}[a] = \{x_i \in \mathcal{O}_i : x_i \cap supp[o[a]] \neq \emptyset\} \tag{5}$$

We denote $supp_i[G, \theta] = supp_{i,(G,\theta)}[\bigcup_{\mu \in supp[\theta]} supp[\mu]]$ as the support of the process from the perspective of player i.

We now introduce a notion of equivalence of strategies that takes into account the information θ of how players choose their strategies.

Definition 4 (Outcome Equivalent Strategies)**.** Two strategies $\mu_i \approx_{G,\theta} \mu'_i$ of player i are outcome equivalent in (G, θ) if $\rho_{G,\theta}|\mu_i[o_i] = \rho_{G,\theta}|\mu'_i[o_i]$ for all $o_i \in \mathcal{O}_i$.

Thus, two strategies are outcome equivalent for player i, if their conditional probability over i's individual outcomes is identical.

Example. In the Montgomery Bus Game, player 1's mixed strategies $\mu_1 = \frac{1}{2}\mathbb{1}_{a_1^1} \oplus \frac{1}{2}\mathbb{1}_{a_1^3}$ and $\mu'_1 = \frac{1}{2}\mathbb{1}_{a_1^2} \oplus \frac{1}{2}\mathbb{1}_{a_1^4}$ are outcome equivalent irrespective of player 2's behavior. Moreover, $\mathbb{1}_{a_1^1} \approx_{G,\theta} \mathbb{1}_{a_1^2}$ if player 2 plays a_2^3 with zero probability in (G, θ). Note that if $\mathcal{O}_2$ partitions the outcomes of the game into the trivial partition $\mathcal{O}_2 = \{\{u, v, w, x, y, z\}\}$, then all strategies of player 2 are outcome equivalent for player 2. *End of example.*

Definition 5 (Outcome Equivalent Processes)**.** Two processes $(G, \theta), (G', \theta')$ are outcome equivalent for player i, $(G, \theta) \approx_i (G', \theta')$, if there exists a bijection

$b_i : (\Delta\mathcal{A}_i)/{\approx_{G,\theta}} \to (\Delta\mathcal{A}'_i)/{\approx_{G',\theta'}}$, such that for all $\mathcal{M}_i \subseteq (\Delta\mathcal{A}_i)/{\approx_{G,\theta}}$: $\theta_i[\mathcal{M}_i] = \theta'_i[b_i[\mathcal{M}_i]]$, and for all $\mu_i \in \mathcal{M}_i, \mu'_i \in b_i[\mathcal{M}_i], x_i \in \mathcal{O}_i$:

$$\rho_{G,\theta}|\mu_i[x_i] = \rho_{G',\theta'}|\mu'_i[x_i]. \tag{6}$$

Two processes $(G,\theta),(G',\theta')$ are outcome equivalent, $(G,\theta) \approx (G',\theta')$, if they are outcome equivalent for all players.

In other words, processes are outcome equivalent if the (equivalence classes of) strategies with the same conditional probability over individual outcomes can be matched such that their total probability according to θ, θ' are identical.

Information is an important aspect in strategic interactions between individuals. Commonly, information sets and subgames are defined in extensive form games. Mailath et al. (1993) show that information sets and subgames can also be defined in reduced game forms in a corresponding manner. Based on their definition of subgames in game forms we now define the corresponding notion of a subprocess of a process. If a game has a subgame, then conditioning the probabilistic beliefs of the policy maker over strategies to that subgame results in a subprocess.

Definition 6 (Subprocess)**.** For a process (G,θ), let $\mathcal{B} = \prod_{i\in\mathcal{N}} \mathcal{B}_i$ be a normal form subgame of G. Then $(G,\theta)|\mathcal{B} = ((\mathcal{N},\mathcal{B},o|_{\mathcal{B}}),\theta')$ defined by

$$\forall i \in \mathcal{N}, \mu_i \in \Delta\mathcal{A}_i : \quad \theta'_i[\mu_i|\mathcal{B}_i] = \frac{\theta_i[\mu_i]\mu_i[\mathcal{B}_i]}{\sum_{\mu'_i} \theta_i[\mu'_i]\mu'_i[\mathcal{B}_i]} \tag{7}$$

is a subprocess of (G,θ) on $\mathcal{B}$.

Example. Suppose player 1 plays either a mixed strategy involving actions a_1^5 and a_1^1 or plays the pure strategy $\mathbb{1}_{a_1^5}$ according to the information of the policy maker. If the policy maker wants to separately analyze the subgame $\{a_1^1,\dots,a_1^4\} \times \{a_2^1,a_2^2\}$, the policy maker has to take into account that the probabilities of the two strategies change; the second strategy never reaches the subgame and therefore the first strategy is played in the subprocess with certainty. Moreover, the mixed strategies must be conditioned on the subgame; in the first strategy, a_1^1 is played with certainty on the subgame. *End of example.*

Using the above definition of Outcome Equivalence, we can define what it means that two processes $(G,\theta),(G,\theta')$ agree outside a subgame $\mathcal{B} \subseteq \mathcal{A}$.

Suppose for some $b \in \mathcal{B}$,

$$o''[a] = \begin{cases} o[a] & a \in \mathcal{A} \backslash \mathcal{B} \\ o[b] & a \in \mathcal{B}. \end{cases} \tag{8}$$

Then the two processes (G, θ), (G, θ') agree outside the subgame $\mathcal{B}$ if $((\mathcal{N}, \mathcal{A}, o''), \theta)$ $\approx ((\mathcal{N}, \mathcal{A}, o''), \theta')$. Therefore two processes agree outside a subgame $\mathcal{B}$ if making all actions on this subgame equivalent yields two outcome equivalent processes.

In some cases, we may want to capture the uncertainty the policy maker faces about what process will arise from the policy. We therefore define mixtures of processes as follows.

Definition 7 (Process Mixture)**.** The mixture of two processes, (G, θ) and (G', θ'), is defined as $\alpha(G, \theta) \otimes (1-\alpha)(G', \theta') = ((\mathcal{N}, \mathcal{A}'', o''), \theta'')$ with

$$\begin{aligned} \mathcal{A}'' &= \mathcal{A} \times \mathcal{A}' \\ \mathcal{A}''_i &= \mathcal{A}_i \times \mathcal{A}'_i \\ o''[a, a'] &= \alpha o[a] \oplus (1-\alpha) o'[a'] \\ \theta''[\mu \otimes \mu'] &= \theta[\mu] \theta'[\mu']. \end{aligned} \tag{9}$$

The mixture weight α represents how likely the policy maker believes it is that the process (G, θ) will be played. We can alternatively interpret this as nature determining which process will be played after the players have determined their strategies in each process independently.

Example. The policy maker may be informed by data that strategies are race-dependent. According to the data, if player 1 is black, she plays the pure strategy $\mathbb{1}_{a_1^1}$ with certainty, if she is white, $\mathbb{1}_{a_1^3}$ instead. Player 2 plays $\mathbb{1}_{a_2^1}$ with certainty. Let the corresponding processes be (G, θ) and (G, θ'). Since behind the veil of ignorance, race is determined by nature's lottery according to some proportion α, the process $\alpha(G, \theta) \otimes (1-\alpha)(G, \theta')$ represents the policy maker's beliefs of the overall process after receiving the information that strategies are race dependent. The game form of this process lets both players choose an action depending on the race of player 1. Player 1 chooses $\mathbb{1}_{(a_1^1, a_1^3)}$ with certainty. The correlation between the race of player 1 and the outcomes is generated by nature which determines the outcomes with probability α as if player 1 had chosen a_1^1 and with probability $1-\alpha$ as if player 2 had chosen a_1^3. *End of example.*

In the axiomatization, we will analyze changes in processes that involve informing the policy maker that a strategic choice was actually determined by nature. In some initial process, the policy maker is informed that some player makes a choice between strategies. Now imagine informing the policy maker with the exact same information, except that according to the new information, the player did not choose among two strategies but instead a random process (nature) made this choice. We call the change from the initial to the latter process a choice removal.

Definition 8 (Choice Removal)**.** Suppose $\mathcal{M}_i \subseteq supp[\theta_i]$ is a set of strategies of player i. Then $D_i^{\mathcal{M}_i}(G,\theta)$ randomizes the choice among these strategies.

$$D_i^{\mathcal{M}_i}(G,\theta) = \bigotimes_{\mu \in \mathcal{M}_i} \frac{\theta[\mu]}{\theta[\mathcal{M}_i]} \left(G, \theta_{-i} \otimes \left(\theta[\mathcal{M}_i]\mathbb{1}_\mu \oplus (1-\theta[\mathcal{M}_i])\theta|\mathcal{M}_i^C\right)\right). \tag{10}$$

If all strategies of a player are randomized, we denote $D_j^{supp[\theta_j]} = D_j$. The choice removal for all players in set $\mathcal{N}' \subseteq \mathcal{N}$ is denoted by $D_{\mathcal{N}'}$ and $D_{-\mathcal{N}'} = D_{\mathcal{N}\setminus\mathcal{N}'}$.

We interpret the above notation as follows. The process inside the brackets is the mixed belief of the policy maker whether a particular strategy $\mu_i \in \mathcal{M}_i$ is being played or a strategy outside $\mathcal{M}_i$ is being chosen. The latter choice among strategies is made with the same probabilities with which the strategies outside $\mathcal{M}_i$ were chosen in (G,θ). Via nature's randomization over processes, nature determines which of these strategies $\mu_i \in \mathcal{M}_i$ is chosen.

Example. We continue the example of a process mixture. Suppose initially, the policy maker has no data on race being the determining factor in the strategic choice of player 1. Instead, the policy maker falsely believes that with probability α, player 1 chooses $\mathbb{1}_{a_1^1}$ and with probability $1-\alpha$, player 1 chooses $\mathbb{1}_{a_1^3}$. In this case, the policy maker attributes the choice therefore to player 1. Let this process be denoted by (G,θ''). Then $D_i(G,\theta'') = \alpha(G,\theta) \otimes (1-\alpha)(G,\theta')$. In other words, the difference between the process in which race determines the choice of the player and the process in which the choice is of player 1's own volition is the choice removal operation. *End of example.*

Some game forms are effectively lotteries from the perspective of some players. Such players have no meaningful strategic choice and therefore removing their strategic choice leaves an outcome equivalent process. We use this idea to capture whether a player is influential or not.

Definition 9 (Influential Players)**.** A player i is *influential* in process $(G, \theta) \in \mathcal{P}$ if $D_i(G, \theta) \not\approx (G, \theta)$. The set of all *influential* players is $\{i \in \mathcal{N} : D_i(G, \theta) \not\approx (G, \theta)\}$.

We conclude this section with a summary of introduced concepts. We defined processes as combinations of game forms with probabilistic information of the policy maker about the strategies players choose. Next, we introduced outcome equivalence as a way to determine similarity of player's strategies across game forms. Further, we introduced subprocesses as the corresponding concept to subgames as introduced in Mailath et al. (1993). Finally, the choice removal removes agency from a player and hands it to nature. Choice removal allows us to express whether a player is influential or not.

4 Axiomatization

We phrase the problem of finding a measure of freedom as a problem of finding a representation of the policy maker's preference relation $\succsim$ over processes $\mathcal{P}$. Behind the veil of ignorance, the policy maker must decide which process to implement for the players and forms preferences according to certain desirable criteria described below. Under these criteria, we then obtain a representation defined as:

Definition 10 (Representation)**.** A function $U : \mathcal{P} \to \mathcal{R}$ represents a binary relation $\succsim$ if for all $a, b \in \mathcal{P}$,

$$a \succsim b \Leftrightarrow U(a) \geq U(b). \tag{11}$$

U is called a representation of $\succsim$.

To ensure that the relation is nontrivial, we employ the following definition of essentiality:

Definition 11. A pair of social outcomes x, y is essential for player i if $\nexists x_i \in \mathcal{O}_i : x, y \in x_i$ and it is not the case that all processes on the set $\{(G, \theta) \in \mathcal{P} : G \in \mathcal{G}[\{x, y\}]\}$ are indifferent.

With the first axiom, we assume the policy maker has a complete and transitive preference relation on processes.

Axiom 1 (Rationality)**.** $\succsim$ is a weak order on $\mathcal{P}$, i.e.,

— $a, b \in \mathcal{P}$ implies $a \succsim b$ or $b \succsim a$ or both.

— $a \succsim b$, $b \succsim c$ imply $a \succsim c$.

Transitivity is a natural requirement from a rationality perspective. However, completeness relies on the policy maker having to rank all possible processes. This is more restrictive as the policy maker may be unwilling to rank two processes on different decision domains. The policy maker may also find some game form G and some information θ incompatible with each other and may therefore be unable to compare (G, θ) to other processes.

Processes that are similar to each other should also be similarly ranked by the policy maker. We therefore adapt two conditions that ensure this. The following Continuity axiom ensures that convergence of information about behavior ensures convergence in preference of the policy maker.

Axiom 2 (Continuity)**.** For all $p \in \mathcal{P}$ and all game forms G the lower and upper sets of $\succsim$, $\{\theta \in \prod_{i=1}^{n} \Delta\Delta\mathcal{A}_i : p \succsim (G, \theta)\}$ and $\{\theta \in \prod_{i=1}^{n} \Delta\Delta\mathcal{A}_i : (G, \theta) \succsim p\}$ are closed.

Continuity requires that for a fixed game form, similar information over players' strategies yields a similar ranking in the preference. It does not require that similar game forms are similarly ranked. For this, we assume the following Outcome Equivalence axiom.

Axiom 3 (Outcome Equivalence)**.** $(G, \theta) \approx (G', \theta') \Rightarrow (G, \theta) \sim (G', \theta')$.

Outcome equivalence makes different game forms comparable. Game forms only matter to the extent that they generate strategic choices with different conditional probabilities over individual outcomes.[9]

Example. In the Montgomery bus game, the policy maker may have some observational data about players' behavior. The policy maker however does not observe the action in which player 1 buys her own bus and drives by herself. Neither does the policy maker observe that player 1 constructs her own vehicle, goes by airplane etc.. However, most likely the policy maker cannot exclude with certainty that these options were not part of the game form when the choice was made. Outcome Equivalence handles this issue by imposing on preferences that changing the game form to allow for such actions does not change the preferences of the policy maker unless these actions

[9]Note that Outcome Equivalence creates large equivalence classes of processes. It may be interesting to consider changes to this axiom to measure other aspects such as power or information transmission in games.

are chosen with positive probability. This is the central advantage of ranking combinations of game forms with information about behavior instead of game forms. Representing policies by game forms would probably also require the choice of the game form to contain implicitly the policy maker's beliefs about which actions are played. Alternatively, one would need to rely on definitions such as what the *legal* or *possible* actions are. In the Montgomery bus game, including only *legal* actions would remove the actually played actions a_2^2 and a_2^3. Allowing for all *possible* actions would add the action of player 1 to construct her own vehicle. Neither of these options seems attractive for the purposes of policy evaluation. *End of example.*

We now impose independence conditions on three levels. First, on lotteries over outcomes, second on the probabilistic information over strategies, and third on subprocesses. The axiom on the independence of lotteries over outcomes is the standard von Neumann-Morgenstern axiom adapted to our setting. Processes in which no player is influential are effectively lotteries and thus we apply the independence axiom with respect to such processes.

Axiom 4 (Lottery Independence)**.** Suppose no player is influential in (G, θ), (G, θ'), (G, θ''), then,

$$(G, \theta) \succsim (G, \theta') \tag{12}$$

$$\Leftrightarrow \quad \alpha(G, \theta) \otimes (1 - \alpha)(G, \theta'') \succsim \alpha(G, \theta') \otimes (1 - \alpha)(G, \theta'') \tag{13}$$

In other words, if nature fully controls the outcomes of the players, then the standard independence axiom holds. In combination with Rationality and Continuity, this axiom requires the decision maker to have expected utility preferences over pure lotteries, i.e., processes in which nature determines the outcomes and players have no meaningful choice.

We emphasize that Lottery Independence is a *weakening* of full independence across all processes. In essence, we are allowing the decision maker to have a strict preference for uncertainty about player's strategies over identical uncertainty over moves of nature but *do not require* the decision maker to do so. Full independence across both types of uncertainty would prevent the decision maker from having such preferences. Thus, arguing in favor of full independence across all processes would require presenting reasons why the policy maker should *never* prefer uncertainty about strategies to uncertainty about nature's moves. Given that the decision maker may believe that selfdetermination of outcomes by players is superior to random assignment by nature, full independence does not seem to be normatively warranted.

Lottery Independence excludes certain value judgments about institutions. Most importantly, it does not allow for source-dependent (Chew & Sagi, 2008) attitudes towards the uncertainty generated by the moves of nature. Source-dependence only arises between uncertainty generated by different players and nature.

Example. Consider nature's lottery over race and gender in the Montgomery Bus Game. A policy maker who finds outcome-dependence on gender undesirable but outcome-dependence on race acceptable violates Lottery Independence. Suppose (G, θ) is a process in which player 1 is a black female and (G, θ') is the process in which player 1 is a black male. In each process, the policy maker believes that player 1 plays a_1^1 with certainty and player 2 plays a_2^1. The policy maker is indifferent between the two processes. Now suppose (G, θ'') is the process of a white male who chooses a_1^3. Then, the LHS mixture of (13) is a gender-and-race-dependent lottery while the RHS is a race-dependent lottery. Since gender-dependent lotteries are intrinsically undesirable to the policy maker, the indifference no longer holds – the policy maker prefers the RHS mixture to the LHS, violating Lottery Independence. *End of example.*

Strategy Independence ensures that the value of choice across strategies is independent across the different strategies.

Axiom 5 (Strategy Independence)**.** Suppose $\theta_i[\mu_i] = \theta'_i[\mu_i]$ and $\rho_{G,\theta}|\mu_i = \rho_{G,\theta'}|\mu_i$ for all $\mu_i \in \mathcal{M}_i \subseteq \Delta A_i$, then

$$(G, \theta) \succsim (G, \theta') \quad \Leftrightarrow \quad D_i^{\mathcal{M}_i}(G, \theta) \succsim D_i^{\mathcal{M}_i}(G, \theta') \tag{14}$$

In other words, the value of choice between two strategies does not depend on the other choices being made. The choice removal $D_i^{\mathcal{M}_i}$ has the effect of taking the choice between some strategies $\mathcal{M}_i$ out of the control of player i. From the perspective of all other players, the game remains unchanged; the probability of each outcome given any of their strategies is the same as before choice removal.

Example. Suppose the policy maker believes that player 1 chooses with equal probability to play the pure strategies $\mathbb{1}_{a_1^1}$, $\mathbb{1}_{a_1^3}$, or $\mathbb{1}_{a_1^5}$. In another process, the policy maker believes that player 1 chooses with equal probability $\mathbb{1}_{a_1^1}$, $\mathbb{1}_{a_1^3}$, or

$\mathbb{1}_{a_1^4}$. Suppose that the policy maker learns that the choice between the pure strategies of $\mathbb{1}_{a_1^1}$, or $\mathbb{1}_{a_1^3}$ is determined by race. Then the ranking between the two processes remains unchanged if the behavior of the other player is identical in both processes. The latter requirement is crucial; in case player 2 plays the pure strategy a_2^3, the choice between $\mathbb{1}_{a_1^1}$ and $\mathbb{1}_{a_1^3}$ is meaningless. In case player 2 plays a_2^1, then the choice of player 1 is effectively between outcomes u_1 and v_1. Strategy Independence therefore captures that ceteris paribus the value of making a strategic choice instead of nature determining the choice does not depend on other strategic choices. *End of example.*

Next, Subprocess Monotonicity ensures that the value of choice across subprocesses is independent if the outcomes of the subgame are independent of the remainder of the game.

Axiom 6 (Subprocess Monotonicity)**.** Let (G, θ) and (G, θ') be equivalent outside the non-null subgame $\mathcal{B}$. Suppose the set of influential players in both processes is $\mathcal{N}'$ and for all $i \in \mathcal{N}'$ we have that $supp_i[o[\mathcal{B}]] \cap supp_i[o[\mathcal{A} \backslash \mathcal{B}]] = \emptyset$. Then,

$$(G, \theta) \succsim (G, \theta') \Leftrightarrow (G, \theta)|\mathcal{B} \succsim (G, \theta')|\mathcal{B}. \qquad (15)$$

Put simply, the relation $\succsim$ is monotone in subprocesses that have a disjoint support from the remainder of the game form: we can improve a process by improving any subprocess unless some of the outcomes of the subprocess are identical to the remainder of the process. The central idea behind this axiom is the following: if a set of players is influential, they can make choices to favor their interests. If they can better favor their interests in a subgame, then this is preferable from the perspective of the policy maker. However, this is only the case if this improvement does not come at the cost of influence across the entire game.

Example. This example illustrates the need for requiring disjoint outcomes. Suppose only player 1 is influential and is choosing between sitting and standing on the bus via the pure strategies $\mathbb{1}_{a_1^1}$ and $\mathbb{1}_{a_1^3}$. Let's suppose the policy maker is indifferent between disallowing standing on the bus or disallowing sitting on the bus (essentially, limiting the player to either of the two actions). Since outcomes are subgames, any game involving a choice between standing and sitting contains the subgame in which the player stands or sits with certainty. Under Subprocess Monotonicity *without* requiring disjoint outcomes, replacing the sitting subgame by the subgame in which the player stands would leave the policy maker indifferent. But then the agent is left in the overall game with a trivial decision between standing and standing. This means that a meaningful choice (between sitting and standing) and a meaningless choice (between standing and standing) are equally good according to the preferences of the policy maker.

The problem can be resolved by requiring disjoint outcomes in Subprocess Monotonicity. Replacing the subgame in which the player sits by a subgame in which the player stands does not leave the policy maker indifferent in case somewhere in the remainder of the process the player stands. The monotonicty axiom in this case does not bind since the outcomes of the two subprocesses are not disjoint. *End of example.*

In the appendix, we prove the following theorem.

Theorem 1. *Suppose for every player i there are at least four essential pairs of outcomes. The relation $\succsim$ on the process space $\mathcal{P}$ fulfills Axioms 1-6 if and only if there exists a continuous, real valued representation $U : \mathcal{P} \to \mathbb{R}$, and for every player i a function $v_i : \mathcal{O}_i \to \mathbb{R}$ and a real number d_i such that*

$$U[G,\theta] = \sum_{i \in \mathcal{N}} U_i[G,\theta] \tag{16}$$

$$U_i[G,\theta] = \sum_{\mu_i} \theta_i[\mu_i] \sum_{x_i \in \mathcal{O}_i} \rho_{G,\theta}|\mu_i[x_i] \left(v_i[x_i] + d_i \ln \left[\frac{\rho_{G,\theta}|\mu_i[x_i]}{\rho_{G,\theta}[x_i]} \right] \right) \tag{17}$$

For each player, the measure consists of an expected valuation of the outcomes and the player's control over own individual outcomes. We call the expected valuation the instrumental value of the process and the control measure the freedom value of the process. We denote the freedom measure of player i by $F_i[G,\theta] = U_i[G,\theta] - U_i[D_i(G,\theta)]$.

The policy maker has for each role in the institution expected utility preferences over implementing specific outcomes. Freedom is measured by the mutual information between strategies and outcomes. Mutual information is a measure of correlation that imposes no structure on the relation between variables. In comparison, the correlation coefficient assumes a linear relation. Spearman's rank order correlation assumes that each of the variables can be ordered. Since the policy maker is given no information about the intention behind strategies, mutual information is the adequate correlation measure for judging the degree to which players use strategies to control outcomes. By separating the mutual information terms into entropies, we can understand the workings of the mutual information measure as follows. The policy maker uses for each player the Shannon entropy to evaluate the quantitative diversity of the outcomes reached (i.e., the Suppes measure). If all outcomes are reached with equal likelihood, then this entropy is maximal and equal to the logarithm of the number of distinct outcomes reached. If only a small amount of outcomes are reached or if few outcomes are reached with high probability, then the quantitative diversity will be low. To account for control, the policy maker deducts the expected conditional entropy of the outcomes given the strategies of the players. Thus, the more quantitative diversity of outcomes is left after choosing a specific strategy, the lower the freedom measure. This quantitative diversity of outcomes can either be due to other players' strategies or due to nature.

Example. We return to the Montgomery bus game to exemplify the data necessary to apply the measure. As stated before, the advantage of our framework is that it does not rely on unobservable information about utilities. Instead, we require observational or experimental data about the strategies of the players that can help inform the policy maker.[10] For simplicity, we focus on the subprocess induced by the subgame $\{a_1^1, \ldots, a_1^4\} \times \{a_2^1, a_2^2\}$. This is meaningful thanks to the Subprocess Monotonicity axiom; if we are interested in the freedom of choice of the overall process, we would simply need to calculate the freedom of choice from the remaining process with an arbitrary outcome u substituted for the subgame in which player 1 boards the bus. To the freedom obtained for this process we then add the sum of the freedom from the subprocess times the probability of reaching this subprocess. More succinctly put, the freedom of an overall process consists of the expected freedom of the disjoint subprocesses plus the freedom derived from determining which of these subprocesses is played. While this is not obvious from the functional form of the freedom measure, it is guaranteed via Subprocess Monotonicity. This decomposability property is important for applications since it allows us to focus on localized data specific to an interaction.

The reduced form of the subgame contains the action profiles $\{\{a_1^1, a_1^2\}, \{a_1^3, a_1^4\}\} \times \{a_2^1, a_2^2\}$. Under the institutional setting of Montgomery in the

[10]The central difficulty of applying the measure to observational data is the possible existence of mixed strategies. In the Montgomery bus game there is no good reason to assume that players play mixed strategies but in other games this might be different. Then observational data of the actions taken is not sufficient for the estimation of freedom of choice since the choice is between mixed strategies, not actions. Instead, the mixed strategies would need to be identified via experimental treatments.

early 1950s, we can inform a policy maker using the following information. To know the weight attached to nature's lottery over race, the policy maker needs to know the ridership composition, or more precisely the frequency with which individuals of different background play the Montgomery bus game. Next, we need to know the strategies chosen by the players, conditional on race. An extremely small fraction of courageous black women took the action $\{a_1^3, a_1^4\}$. Since in their cases the driver took action a_2^2, they were arrested. The overwhelming majority of black passengers endured the discriminatory treatment by taking action $\{a_1^1, a_1^2\}$. To account for cases in which player 1 refused to yield their seat and managed to keep the seat, we would also need to obtain information whether any drivers took action a_2^1 when interacting with a black passenger. Lacking evidence of such cases, we assume that this did not happen. Similarly, there are no known accounts of white passengers being arrested for refusing to give up their seat. To estimate their freedom of choice of voluntarily yielding their seat, we need to know the fraction of white passengers that take action a_1^1. If α is the fraction of black passengers, ϵ the fraction of black passengers refusing to yield their seat and being arrested, and γ the fraction of white passengers yielding their seat voluntarily, then the non-instrumental component of freedom of choice yields after rearranging terms:

$$
\begin{aligned}
& (U_1[G,\theta] - U_1[D_1(G,\theta)])/d_i \\
= \quad & \epsilon \cdot \alpha \left(\ln \frac{1}{\epsilon} \right) \\
& + (1-\gamma)(1-\alpha) \left(\ln \frac{1}{1-\gamma} \right) \\
& + (\alpha(1-\epsilon) + (1-\alpha)\gamma) \left(\ln \frac{1}{\alpha(1-\epsilon)+(1-\alpha)\gamma} \right) \\
& + \epsilon\gamma(1-\alpha)\ln(1-\alpha) + (1-\epsilon)(1-\gamma)\alpha \ln \alpha \qquad (18)
\end{aligned}
$$

The measure is monotonically increasing in ϵ for the given scenario in which ϵ is small. A crucial point is that the freedom of player 1 is not the weighted sum of the freedom of a white player 1 and a black player 1. The fact that the outcomes are partially determined by nature's lottery over race is intrinsically undesirable which prevents this separability. In the following, we interpret the terms in (18). The first three rows of (18) are the entropy of the outcomes reached. In case player 1 would have perfect control over which of the outcomes

u, v, w arises, this would be the freedom of choice of the player.[11] However, the outcome partially depends on nature's lottery. The last row of (18) corrects for this with two negative terms. The first term corrects for the probability $1 - \alpha$ with which player 1 reaches outcome u when "choosing" outcome w. The second term corrects for the probability with which player 1 reaches w when "choosing" u. Both terms are negative since α and $1 - \alpha$ are smaller than one. In each case, the correction term arises from the fact that the conditional probability of the outcome given the strategy is not equal to one.

The example also shows that the freedom measure is completely neutral towards the characteristic of the outcomes. All judgments regarding whether for example the outcome v is in the eyes of the policy maker more desirable than the outcome w can only be contained in the instrumental value of the process. *End of example.*

It is important not to confuse v_i with the utility representations of individuals. v_i instead represents the policy maker's value for implementing specific outcomes for players and cannot contain players' utilities. Since utility is not observable, it must be estimated from behavior. But the axioms prescribe that behavior only enters the mutual information measure in a meaningful way. This has far reaching consequences for a utilitarian policy maker in practice. Either the policy maker accepts that in the absence of cardinal utility information the mutual information criterion is the best we can do to *approximate* utilitarianism or the policy maker must argue why any of the stated axioms must be violated in the estimation of the expected utility of every player.

A commonly employed assumption in the freedom of choice literature removes the dependence of freedom of choice on the policy maker's norms by assuming that all singletons are indifferent. We can translate this condition into our setting as saying that all outcomes are instrumentally equally valuable to the policy maker:

Remark 1. Suppose we assume the assumption of the indifference of no-choice axiom (Pattanaik & Xu, 1990), i.e., for all social outcomes x, y, the trivial games yielding the outcomes with certainty are indifferent, $((\mathcal{N}, \{a\}, a \mapsto x), \mathbb{1}_{\mathbb{1}_a}) \sim ((\mathcal{N}, \{a\}, a \mapsto y), \mathbb{1}_{\mathbb{1}_a})$. Then the instrumental value is a constant and U only depends on the mutual information between strategies and individual outcomes of every player.

[11] Indeed, this would be the Suppes (1996) measure.

We now discuss under which additional assumptions and domain restrictions the proposed freedom measure becomes identical to various measures of the literature. For an overview, see Figure 2. Naturally, since the policy maker's preferences in this paper are on a richer domain, this does not imply that the representation theorem is a generalization of previous characterizations. Instead, it is a particular extension of the ordering of other measures to a richer domain.

Suppes (1996) proposes to measure freedom as the entropy of the relative frequencies with which an agent chooses the alternatives of an opportunity set:

Definition 12. Entropy Freedom Measure (Suppes, 1996)
$F_S(P) = -\sum_{x \in \mathcal{C}} P(x) \ln P(x)$ where $P(x)$ refers to the probability with which an agent chooses element x of the opportunity set $\mathcal{C}$.

The entropy measure increases both in the total number of options chosen with positive probability and how even the distribution of these chosen outcomes is.

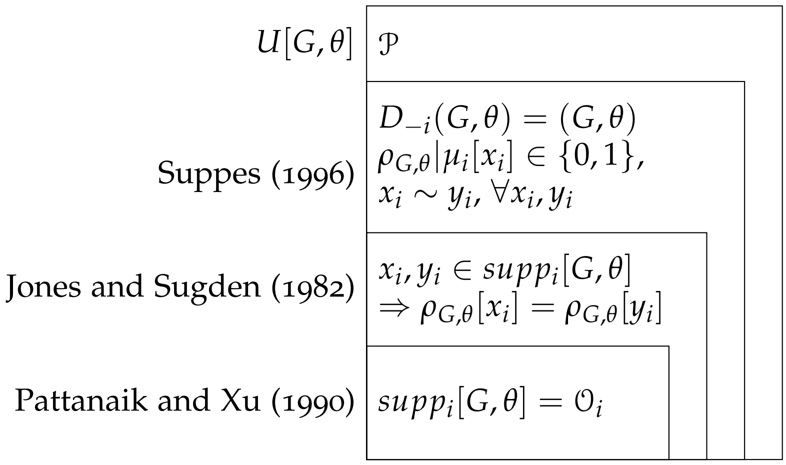

Figure 2: Relation of Freedom Measures

Since the Shannon entropy is a limiting case of mutual information, our measure extends the measure of freedom by (Suppes, 1996) in a natural manner:

Remark 2. Suppose indifference of no-choice holds and only player i is influential. Moreover, let $supp(\theta_i)$ contain only pure strategies and for all a, $supp[o[a]] \subseteq x_i$ for some $x_i \in \mathcal{O}_i$. Then, $U[D_{-i}(G,\theta)] = F_S((x \mapsto x_i)\sharp\rho_{G,\theta})$.

This establishes the relation of the measure to the Suppes measure. In case an individual has perfect control over outcomes and the policy maker is indifferent between all outcomes, then the measure is equal to the Suppes measure of the distribution of the probability distribution over outcomes. The result follows from the fact that $supp[o[a]] \subseteq x_i$ ensures $\rho|\mathbb{1}_a[x_i] = 1$. From this result, the comparisons to other measures directly follow. Since the Suppes measure extends the ranking of several other measures (for details, see Appendix), the above remark also establishes the relation to various other measures. Figure 2 displays the relation between different measures.

5 Concluding Remarks

The policy evaluation criterion we presented is consistent with three principles that are commonly employed in economics. First, the criterion only depends on observable information; in classical welfare economics this information used is ordinal preference, in our game theoretic setting, the information used is the strategy of a player. Second, based on the available information, the criterion defaults to maximizing the control of individuals. Third, in game forms the criterion obeys independence across (disjoint) subgames. Without this independence, we would need to worry that improving freedom of choice in some context negatively impacts overall freedom of choice. Taken together, these principles guarantee that the the criterion can readily be applied[12] to many contexts, two of which have been exemplified in this paper.

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[12]A variant of the measure has been applied to individual's preference for decision rights in Neri and Rommeswinkel (2014). Rommeswinkel and Wu (2020) applies the measure to the choice of a central bank constitution. Chen and Rommeswinkel (2020) uses consumption data to estimate freedom of choice of U.S. consumers.

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A Proof of Theorem 1

The overall proof structure is as follows:

1. We first prove some technical lemmas that are useful. These include connectedness of the order topology on the set of processes and a result that allows us to deduce a quasi-separable representation of the form $f(x,z) + g(y,z)$ from two conditionally additive representations of the form $h(f(x,z) + g(y,z), z)$. The key axioms used are Continuity and Outcome Equivalence.

2. Next, we show that on the space of processes that are lotteries (i.e., in which no player other than nature is influential), we have an expected utility representation. The key axioms used are Lottery Independence and Outcome Equivalence.

3. Following, we prove that there exists a representation that is quasi-separable across players and their strategies conditional on the outcome probabilities. More specifically, we show that this representation is linear in the probabilities of strategies. The key axioms used are Strategy Independence, Outcome Equivalence and the expected utility representation over lotteries.

4. Having obtained quasi-separability across all players we can focus on processes that have only a single influential player. We show that the preferences over these processes are additively separable across outcomes. All assumptions except Strategy Independence are used in this step.

5. Next, we combine the linear representation across strategies with the additive representation across outcomes.

6. Lastly, we employ the fundamental equation of the theory of information to solve for the procedural component. The key axiom used is Subprocess Monotonicity and the additive separability of the procedural preferences across strategies and from the expected utility of the subprocesses.

To state each Lemma concisely, we omit repeating the axioms employed in the theorem.

A.1 Technical Lemmas

We define the order topology on $\mathcal{P}$ as the topology generated by the intersections of sets of the form $\{p \in \mathcal{P} : p \succ p'\}$ and $\{p \in \mathcal{P} : p'' \succ p\}$ for arbitrary $p', p'' \in \mathcal{P}$.

Lemma 1. *$\mathcal{P}$ is connected in the order topology.*

Proof. By connectedness of the real numbers and Continuity, the order topology on any subspace of $\mathcal{P}$ of the form $\{(G,\theta) : \theta = \alpha\theta' \oplus (1-\alpha)\theta''\}$ is connected. By completeness and transitivity of the relation, this topology is identical to the subspace topology of the order topology on $\mathcal{P}$. If $\mathcal{P}$ is not connected, then it is the union of two nonempty disjoint open sets $\mathcal{P}'$ and $\mathcal{P}''$. Take any element $p' \in \mathcal{P}'$ and $p'' \in \mathcal{P}''$. The order topology on $\mathcal{P}''' = \{(G,\theta) : \theta = \alpha\theta' \oplus (1-\alpha)\theta''\}$ is disconnected by the nonempty open sets $\mathcal{P}' \cap \mathcal{P}'''$ and $\mathcal{P}'' \cap \mathcal{P}'''$, yielding a contradiction. Thus, $\mathcal{P}$ is connected. □

Lemma 2. *Suppose $f(g(x,y),z) = xa(y,z) + b(y,z)$ holds for continuous functions $f : \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}$ on some sets $\mathcal{Y}$, $\mathcal{Z}$. Let f and g be invertible in the first argument, then,*

$$\begin{aligned} f(r,z) &= h^{-1}(r)j(z) + k(z) \\ g(x,y) &= h(xl(y) + m(y)). \end{aligned} \tag{19}$$

Proof. Let f^{-1}, g^{-1} denote the inverses of f and g in their first arguments, respectively. We use invertibility of the two functions to derive:

$$\begin{aligned} g(x,y) =& f^{-1}(xa(y,z_0) + b(y,z_0), z_0) \\ f(r,z) =& g^{-1}(r,y_0)a(y_0,z) + b(y_0,z) \\ f(g(x,y),z) =& g^{-1}(g(x,y),y_0)a(y_0,z) + b(y_0,z) \\ =& g^{-1}(f^{-1}(xa(y,z_0) + b(y,z_0),z_0),y_0)a(y_0,z) + b(y_0,z) \end{aligned} \tag{20}$$

which, by the assumption that f and g are continuous, is affine in x if and only if $g^{-1}(r,y_0)$ and $f^{-1}(r,z_0)$ are (up to an affine transformation) inverses to each other. The result then follows by appropriate definitions for h, j, k, l, m. □

A.2 Expected Utility Representation on Lotteries

Lemma 3 (Expected Utility on Lotteries)**.** *On the set of nature's lotteries over outcomes, $\{(G,\theta) : (G,\theta) = D_{\mathcal{N}}(G,\theta)\}$, the relation $\succsim$ can be represented by*

$U[G,\theta] = \sum_{o \in \mathcal{O}} \rho[o] U[o]$.

Remark 3. $U[x]$ is shorthand for $U[(\mathcal{N}, \{a\}, a \mapsto \mathbb{1}_x), \mathbb{1}_{\mathbb{1}_a}]$.

Proof. We show that $\succsim$ fulfills the assumptions of Herstein and Milnor (1953). By Outcome Equivalence,

$$D_{\mathcal{N}}(G,\theta) = (G,\theta) \Rightarrow (G,\theta) \approx ((\mathcal{N}, \{a\}, a \mapsto \rho_{G,\theta}), \mathbb{1}_{\mathbb{1}_a}) \tag{21}$$

That is, any process in which no player is influential is outcome equivalent to a trivial process with a single action profile a in which a lottery over outcomes is resolved with probabilities $\rho_{G,\theta}$. The set of probability distributions form a mixture space. Furthermore,

$$\alpha(G,\theta) \oplus (1-\alpha)(G',\theta') \approx ((\mathcal{N}, \{a\}, a \mapsto \alpha\rho_{G,\theta} \oplus (1-\alpha)\rho_{G',\theta'}), \mathbb{1}_{\mathbb{1}_a}) \tag{22}$$

and therefore mixtures between processes translate into mixtures between outcome probability distributions. It follows from Lottery Independence that:

$$\begin{array}{ccccc} & ((\mathcal{N}, \{a\}, a \mapsto \rho_{G,\theta}), \mathbb{1}_{\mathbb{1}_a}) & \succsim & ((\mathcal{N}, \{a\}, a \mapsto \rho_{G',\theta'}), \mathbb{1}_{\mathbb{1}_a}) \\ & \approx & & \approx \\ & (G,\theta) & \succsim & (G',\theta') \\ & = & & = \\ & D_{\mathcal{N}}(G,\theta) & \succsim & D_{\mathcal{N}}(G',\theta') \\ \Leftrightarrow & \alpha D_{\mathcal{N}}(G,\theta) \oplus (1-\alpha) D_{\mathcal{N}}(G'',\theta'') & \succsim & \alpha D_{\mathcal{N}}(G',\theta') \oplus (1-\alpha) D_{\mathcal{N}}(G'',\theta'') \\ & \approx & & \approx \\ & ((\mathcal{N}, \{a\}, a \mapsto \alpha\rho_{G,\theta} \oplus (1-\alpha)\rho_{G'',\theta''}), \mathbb{1}_{\mathbb{1}_a}) & \succsim & ((\mathcal{N}, \{a\}, a \mapsto \alpha\rho_{G',\theta'} \oplus (1-\alpha)\rho_{G'',\theta''}), \mathbb{1}_{\mathbb{1}_a}) \end{array} \tag{23}$$

It follows that on the set of lotteries, $\succsim$ fulfills the independence axiom (Herstein & Milnor, 1953, Axiom 3) with respect to the outcome probabilities $\rho_{G,\theta}$. Rationality and Continuity guarantee their Axioms 1 and 2. The existence of an expected utility representation follows from their Theorem 8. □

A.3 Conditional Linearity in Probabilities of Strategies

Lemma 4 (Separability in Strategies)**.** *There exists a representation of the form:*

$$U[G,\theta] = \sum_{i \in \mathcal{N}} \sum_{\mu_i} \theta_i[\mu_i] v_i(\rho_{G,\theta} | \mu_i, \rho_{G,\theta}) \equiv \sum_{i \in \mathcal{N}} U_i[G,\theta]. \tag{24}$$

Proof. Both the information over strategies $\theta = \prod_{i \in \mathcal{N}} \theta_i$ and the information over strategies of particular individuals θ_i are probability distributions and therefore elements of mixture spaces. We first use Strategy Independence to derive a conditional expected utility representation for each θ_i. We derive the following conditional independence property. If for all $j \in \mathcal{N} - \{i\}$ and all $\mu_j \in \Delta A_j$:

$$\rho_{G,\theta_i \otimes \theta_{-i}} | \mu_j = \rho_{G,\theta'_i \otimes \theta'_{-i}} | \mu_j = \rho_{G,\theta''_i \otimes \theta''_{-i}} | \mu_j \tag{25}$$

$$\rho_{G,\theta_i \otimes \theta_{-i}} = \rho_{G,\theta'_i \otimes \theta'_{-i}} = \rho_{G,\theta''_i \otimes \theta''_{-i}} \tag{26}$$

then

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \tag{27}$$

$$\Leftrightarrow \quad (G, \theta_i \otimes \theta_{-i}) \succsim (G, \theta'_i \otimes \theta_{-i}) \tag{28}$$

We emphasize at this point that the mixture in the above processes each represents uncertainty of the policy maker about the strategies played by the player, not a random choice by nature. The proof of the above independence result uses Strategy Independence and Outcome Equivalence. First we remove choice over strategies in $\mathcal{M}_i = supp[\theta''_i]$. Note that for this purpose, we may assume that θ''_i has disjoint support from θ_i and θ'_i since by Continuity we can choose a disjoint support that is arbitrarily close to the actual support of θ''_i. Applying the choice removal $D^{\mathcal{M}_i}$ on both sides leaves the preference unchanged. Using an outcome equivalent transformation, we have:

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \tag{29}$$

$$\Leftrightarrow \quad (G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu^*_i}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\mathbb{1}_{\mu^*_i}) \otimes \theta_{-i}) \tag{30}$$

where $\mathbb{1}_{\mu^*_i}$ denotes a mixed strategy in which i plays the actions with the same probability with which they are played in θ''_i. Now suppose the marginal distributions fulfill (25). Then there exists some outcome equivalent transformation such that

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu^*_i}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\mathbb{1}_{\mu^*_i}) \otimes \theta_{-i}) \tag{31}$$

$$\Leftrightarrow \quad (G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu^{**}_i}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\mathbb{1}_{\mu^{**}_i}) \otimes \theta_{-i}) \tag{32}$$

where $\mathbb{1}_{\mu_i^{**}}$ plays each action with the same probability as the marginal probability in θ_i. It follows that:

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{**}}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta_i' \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{**}}) \otimes \theta_{-i}) \tag{33}$$

$$\Leftrightarrow \quad (G, \theta_i \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta_i' \oplus \frac{1}{2}\theta_i) \otimes \theta_{-i}) \tag{34}$$

Proceeding in a similar manner we can derive

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{***}}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta_i' \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{***}}) \otimes \theta_{-i}) \tag{35}$$

$$\Leftrightarrow \quad (G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta_i') \otimes \theta_{-i}) \succsim (G, \theta_i' \otimes \theta_{-i}) \tag{36}$$

where $\mathbb{1}_{\mu_i^{***}}$ plays each action with the same probability as in θ_i'.

Combining (34) with (36), we have by transitivity the desired result (27). By Rationality and Continuity, using Theorem 8 of Herstein and Milnor (1953), it follows that for fixed conditional outcome probabilities given the other player's strategies, we have an expected utility representation on θ_i.[13]

Next, it holds for $\theta_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'} = \prod_{i \in \mathcal{N}'} \theta_i \otimes \prod_{i \in \mathcal{N}-\mathcal{N}'} \theta_i$ that if for all $j \in \mathcal{N} - \mathcal{N}'$ and all $\mu_j \in \Delta A_j$:

$$\rho_{G,\theta_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}}|\mu_j = \rho_{G,\theta'_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}}|\mu_j \tag{37}$$

$$= \rho_{G,\theta_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}}|\mu_j \tag{38}$$

$$= \rho_{G,\theta'_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}}|\mu_j, \tag{39}$$

$$\rho_{G,\theta_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}} = \rho_{G,\theta'_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}} = \rho_{G,\theta_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}} = \rho_{G,\theta'_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}}, \tag{40}$$

then:

$$(G, \theta_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}) \succsim (G, \theta'_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}) \tag{41}$$

$$\Leftrightarrow (G, \theta_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}) \succsim (G, \theta'_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}) \tag{42}$$

The proof is almost identical to the above, except that instead of removing choice over the strategies in $\mathcal{M}_i$ for a single player i, instead the choice over the entire strategies ΔA_j is removed for all $j \in \mathcal{N} - \mathcal{N}'$. We therefore have an additively separable representation across the probabilities of the strategies of

[13] Although this result holds for arbitrary i, this of course does not yet imply that (for fixed probability of o) the aggregation (across i) of the expected utility representations is additive.

each of the players.

Finally, we show that the expected utility representation and the additive representation across players are jointly additive. If for some individuals i, j,

$$\rho_{G,\theta_{ij}\otimes\theta_{-ij}}|\mu_k = \rho_{G,\theta'_{ij}\otimes\theta_{-ij}}|\mu_k \tag{43}$$

$$= \rho_{G,\theta_{ij}\otimes\theta'_{-ij}}|\mu_k \tag{44}$$

$$= \rho_{G,\theta'_{ij}\otimes\theta_{-ij}}|\mu_k, \quad \mu_k \in \Delta A_k, \forall k \in \mathcal{N} - \{i, j\}, \tag{45}$$

$$\rho_{G,\theta_{ij}\otimes\theta_{-ij}} = \rho_{G,\theta'_{ij}\otimes\theta_{-ij}} = \rho_{G,\theta_{ij}\otimes\theta'_{-ij}} = \rho_{G,\theta'_{ij}\otimes\theta'_{-ij}} \tag{46}$$

then,

$$(G, \left(\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta'_i\right) \otimes \left(\frac{1}{2}\theta_j \oplus \frac{1}{2}\theta'_j\right) \otimes \theta_{-ij}) \succsim (G, \left(\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta'_i\right) \otimes \left(\frac{1}{2}\theta_j \oplus \frac{1}{2}\theta'_j\right) \otimes \theta_{-ij}) \tag{47}$$

$$\Leftrightarrow (G, \left(\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i\right) \otimes \left(\frac{1}{2}\theta_j \oplus \frac{1}{2}\theta''_j\right) \otimes \theta_{-ij}) \succsim (G, \left(\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i\right) \otimes \left(\frac{1}{2}\theta_j \oplus \frac{1}{2}\theta''_j\right) \otimes \theta_{-ij}) \tag{48}$$

When fixing all θ_k, ρ, and all $\rho|\mu_k$ for all individuals $k \neq i, j$ and strategies μ_k in the support, we can therefore find an additive representation of the form:

$$U[G, \left(\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta'_i\right) \otimes \left(\frac{1}{2}\theta_j \oplus \frac{1}{2}\theta'_j\right) \otimes \theta_{-ij}] \tag{49}$$

$$= f_i[\theta_i] + g_i[\theta'_i] + f_j[\theta_j] + g_j[\theta'_j] \tag{50}$$

Since for the probabilities of strategic choice of each individual we have an expected utility representation, we indeed have:

$$U[G, \left(\alpha\theta_i \oplus (1-\alpha)\theta'_i\right) \otimes \left(\beta\theta_j \oplus (1-\beta)\theta'_j\right) \otimes \theta_{-ij}] \tag{51}$$

$$= \alpha h_i[\theta_i] + (1-\alpha)h_i[\theta'_i] + \beta h_j[\theta_j] + (1-\beta)h_j[\theta'_j] \tag{52}$$

since the expected utility representation is additive and uniqueness of additive representations applies. Further, we can derive $h_i[\theta_i] = \sum_{\mu_i} \theta_i[\mu_i] w[\mu_i]$ using Cauchy's functional equation. By Outcome Equivalence, increasing the probability that μ_i will be played instead of μ'_i only matter if $\rho_{G,\theta}|\mu_i \neq \rho_{G,\theta}|\mu'_i$, thus: $h_i[\theta_i] = \sum_{\mu_i} \theta_i[\mu_i] v_i[\rho_{G,\theta}|\mu_i]]$. While we have only shown additive separability of the expected utility representations for i and j, the extension to n individuals is straightforward and we therefore obtain a representation:

$$U[G,\theta] = V\left[\sum_i \sum_{\mu_i} \theta_i[\mu_i] v_i[\rho_{G,\theta}|\mu_i], \rho_{G,\theta}], \rho_{G,\theta}\right] \tag{53}$$

for arbitrary $\rho_{G,\theta}$.

What is left to show is that V is affine in its first argument. For this, we assume there are three influential players.[14] Consider a process θ such that:

$$\theta = \theta_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}} \otimes \theta_{\mathcal{C}} \otimes \theta_{\mathcal{D}} \tag{54}$$

$$\theta_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}} = \theta_1 \otimes \prod_{j \neq 1} \mathbb{1}_{d_j} \tag{55}$$

$$\theta_{\mathcal{B}} = \theta_2 \otimes \prod_{j \neq 2} \mathbb{1}_{b_j} \tag{56}$$

$$\theta_{\mathcal{C}} = \theta_3 \otimes \prod_{j \neq 3} \mathbb{1}_{c_j} \tag{57}$$

$$\theta_{\mathcal{D}} = \theta_4 \otimes \prod_{j \neq 4} \mathbb{1}_{d_j} \tag{58}$$

This process has three subprocesses in which on each subprocess only a single player chooses between strategies. All other players play a single pure strategy. Player 1's strategies determine which of the subprocesses is being played. We assume that the three subprocesses each have two disjoint outcomes from the remainder of the game form. We have the representation:

$$V\Bigg[\sum_{\mu_1} \theta_1[\mu_1] v_1[\rho_{G,\theta}|\mu_1, \rho_{G,\theta}] \tag{59}$$

$$+ \sum_{\mu_2} \theta_2[\mu_2] v_2[\rho_{G,\theta}|\mu_2, \rho_{G,\theta}] \tag{60}$$

$$+ \sum_{\mu_3} \theta_3[\mu_3] v_3[\rho_{G,\theta}|\mu_3, \rho_{G,\theta}] \tag{61}$$

$$+ \sum_{\mu_4} \theta_4[\mu_4] v_4[\rho_{G,\theta}|\mu_4, \rho_{G,\theta}], \rho_{G,\theta} \Bigg] \tag{62}$$

For fixed outcome probabilities, this representation is additively separable in the three subprocesses. Note that by Subprocess Monotonicity, on the space of processes of the above form, $\succsim$ fulfills joint independence across subprocesses for fixed θ_1. By Gorman (1968) there then exists an additively

[14]It is straightforward to adapt the proof to a single influential player. Employing distinct influential players for each subprocess makes the proof notationally clearer, however.

separable representation of the form:

$$W[f_2[\theta_2, \theta_1] + f_3[\theta_3, \theta_1] + f_4[\theta_4, \theta_1], \theta_1] \tag{63}$$

where W is monotone and thus invertible in its first argument. We show that W must be affine. Indeed, by the existence of an expected utility representation in case $\theta_1 = \mathbb{1}_{\mu_1}, \ldots, \theta_4 = \mathbb{1}_{\mu_4}$ it follows that W is affine if $\theta_1 = \mathbb{1}_{\mu_1}$. Without loss of generality then,

$$V\Bigg[v_1[\rho_{G,\theta}|\mu_1, \rho_{G,\theta}] \tag{64}$$

$$+ \sum_{\mu_2} \theta_2[\mu_2] v_2[\rho_{G,\theta}|\mu_2, \rho_{G,\theta}] \tag{65}$$

$$+ \sum_{\mu_3} \theta_3[\mu_3] v_3[\rho_{G,\theta}|\mu_3, \rho_{G,\theta}] \tag{66}$$

$$+ \sum_{\mu_4} \theta_4[\mu_4] v_4[\rho_{G,\theta}|\mu_4, \rho_{G,\theta}], \rho_{G,\theta}\Bigg] = f_2[\theta_2, \theta_1] + f_3[\theta_3, \theta_1] + f_4[\theta_4, \theta_1] \tag{67}$$

if $\theta_1 = \mathbb{1}_{\mu_1}$. Since the three players 2, 3, 4 are influential, their sum components on the LHS vary for fixed $\rho_{G,\theta}$. It follows that for fixed $\rho_{G,\theta}$, both the RHS and the first argument of V are additive representations across $\theta_2, \ldots, \theta_4$. We therefore obtain by uniqueness of additive representations that V is affine in the first argument. Since $\sum_{\mu_i} \theta_i[\mu_i] = 1$, it is without loss of generality to assume that V is the unit transformation. We therefore obtain the desired representation:

$$U[G, \theta] = \sum_{i \in \mathcal{N}} \sum_{\mu_i} \theta[\mu_i] v_i[\rho_{G,\theta}|\mu_i, \rho_{G,\theta}] \tag{68}$$

$$= \sum_{i \in \mathcal{N}} U_i[G, \theta] \tag{69}$$

□

A.4 Additive Separability on Subprocesses

The remainder of the proof is about specifying the functional form of U_i. In order to identify v_i for some player i, we need to consider only processes in which a single player i is influential, since

$$
\begin{aligned}
& U_i[G,\theta] && (70)\\
=& U[D_{-i}(G,\theta)] - \sum_{j\neq i} U_j[D_{-i}(G,\theta)] && (71)\\
=& U[D_{-i}(G,\theta)] - \sum_{j\neq i} U[D_{-j}D_{-i}(G,\theta)] + \sum_{j\neq i}\sum_{k\neq j} U_k[D_{-j}D_{-i}(G,\theta)] && (72)\\
=& U[D_{-i}(G,\theta)] - U[D_{\mathcal{N}}(G,\theta)] + U_i[D_{\mathcal{N}}(G,\theta)]. && (73)
\end{aligned}
$$

The latter two terms of the expressions have already been determined as expected utility representations. We therefore focus on processes of the form $D_{-i}(G,\theta)$. Using Outcome Equivalence, we can further focus on the processes in which all uncertainty is resolved in the mixed strategies of player i instead of by nature. For this, we define a game form $G^* = (\mathcal{N}, \mathcal{A}^*, o^*)$ such that for some bijection $f : \mathcal{O} \to \mathcal{A}^*$, $o[f(x)] = \mathbb{1}_x$.

To state the following lemma, it is useful to define pushforward measures. If f is a measurable mapping from $\mathcal{S}$ to $\mathcal{S}'$, then $f\#\mu$ is the pushforward measure fulfilling $f\#\mu[s] = \mu[f^{-1}[s]]$ for all $s \in \mathcal{S}'$.

Lemma 5 (Equivalence to game form without nature)**.** *Suppose that* $\theta[\mu] = \theta^*[f\#\bigoplus_a \mu[a]o[a]]$ *for all* $\mu \in supp[\theta]$*, then* $(G,\theta) \sim (G^*,\theta^*)$*.*

Proof. The two processes are outcome equivalent. □

Lemma 6 (Power Removal Lemma)**.** *Let* (G,θ) *be in outcome form with mapping* $f : \mathcal{O} \to \mathcal{A}^*$*. Let* $g : \mathcal{O} \to \prod_i \mathcal{O}_i$ *be the product mapping of each of the canonical maps* g_i *of the partition* $\mathcal{O}_i$*. Let* $\mathcal{A}' = \{\mathcal{B} \subseteq \mathcal{A} : \exists x_i \in \mathcal{O}_i : f[x_i] = \mathcal{B}\}$*. Let* θ' *and* o' *fulfill for all* $a' \in \mathcal{A}'$ *and all* μ*:*

$$o'[a'] = \mathbb{1}_{f^{-1}[a']} \otimes \prod_{j\neq i} g_j\#\rho_{G,\theta} \tag{74}$$

$$\theta'[f \circ g_i \circ f^{-1}\#\mu] = \theta[\mu] \tag{75}$$

Then $(G,\theta) \sim (G',\theta')$*.*

Thus, we replace each action profile in $\mathcal{A}$ with an action that yields a particular outcome from $\mathcal{O}_i$ and the same lottery across all other outcomes. This naturally generates an outcome equivalent process.

Proof. We show that the processes are outcome equivalent. For all $j \neq i$ and all $a' \in \mathcal{A}'$,, $\mu_i \in \Delta\mathcal{A}_i$: $g_j\#\rho_{G,\theta} = g_j\#o'[a'] = g_j\#\rho_{G',\theta'}|\mu_i = g_j\#\rho_{G',\theta'}$. For player i, we first note that $f \circ g_i \circ f^{-1}$ is the canonical mapping from $\mathcal{A}$ to $\mathcal{A}'$, since $\mathcal{A}'$ is a partition of $\mathcal{A}$. All actions in a' yield the same outcome for i as each $a \in a'$. Therefore, $g_i \# \rho_{G,\theta}|\mu = g_i \# \rho_{G',\theta'}|f \circ g_i \circ f^{-1} \# \mu$. □

Note that in processes in which power has been removed, there exists a bijection $f' : \mathcal{O}_i \to \mathcal{A}$ such that the action a of player i determines the outcome $f^{-1}[a]$ with certainty.

To use Subprocess Monotonicity, we require disjoint subgames. Consider the canonical mapping $g : \mathcal{O} \to \mathcal{O}_i$ that maps each element to their equivalence class. Note that the image $f[g^{-1}[x_i]]$ is a subgame of G^*. We now create for every process (G^*, θ) an indifferent process (G^*, θ^*) such that $\mu[f[x]] = \mu[f[x_i]] \frac{\bigoplus_{\mu'} \theta[\mu']\mu'[x]}{\bigoplus_{\mu'} \theta[\mu']\mu'[x_i]}$ for all $\mu \in supp[\theta^*]$. In other words, while strategies may differ about the probabilities of elements of $\mathcal{O}_i$, conditional on reaching $x_i \in \mathcal{O}_i$, all strategies yield the same probability distribution over outcomes. This means that player i can only influence the own outcomes. To prove this, we must first find a way to decompose a process into subgames. This is done in the following Lemma.

Lemma 7 (Disjoint Subprocess Decomposition Lemma). *Let (G, θ) be in powerless outcome form with mapping $f : \mathcal{O} \to \mathcal{A}$. Consider a partition $\overline{\mathcal{A}}$ of $\mathcal{A}$. For any $\mathcal{B} \in \overline{\mathcal{A}}$, denote by $\theta_{\mathcal{B}}$ the conditional probabilities fulfilling $(G, \theta_{\mathcal{B}}) = (G, \theta)|\mathcal{B}$. Let the function $h : supp[\theta] \times \prod_{\mathcal{B} \in \overline{\mathcal{B}}} supp_{\theta_{\mathcal{B}}} \to \Delta\mathcal{A}$ map strategies on every subgame to strategies in the game such that: Define,*

$$h[k\#\mu, \{\mu_{\mathcal{B}}\}_{\mathcal{B}}][\mathcal{C}] = \sum_{\mathcal{B}} k\#\mu[\mathcal{B}]\mu_{\mathcal{B}}[\mathcal{B} \cap \mathcal{C}] \tag{76}$$

$$\theta^*[h[\mu', \{\mu_{\mathcal{B}}\}]] = \theta[\mu'] \prod_{\mathcal{B}} \theta_{\mathcal{B}}[\mu_{\mathcal{B}}] \tag{77}$$

$$b : \left(k\#\mu, \{\mu|\mathcal{B}\}_{\mathcal{B} \in \overline{\mathcal{A}}}\right) \mapsto \bigoplus_{\mathcal{B}} k\#\mu[\mathcal{B}]\mu|\mathcal{B} \tag{78}$$

$$\theta_{\overline{\mathcal{A}}}[k\#\mu] = \theta[\mu] \tag{79}$$

$$\theta^* = b\# \left(\theta_{\overline{\mathcal{A}}} \otimes \prod_{\mathcal{B}} \theta_{\mathcal{B}}\right). \tag{80}$$

Then, $(G^*, \theta) \sim (G^*, \theta^*)$.

Proof. The two processes are identical on every subprocess. By subprocess monotonicity, if after equalizing the outcomes within each of the subprocesses the two processes are equivalent, then they are indifferent. Note that after identifying the outcomes of the actions within each subprocess, all strategies μ that are identical under the pushforward $k\#\mu$ are strategically equivalent. It follows that in this case every $\mu \in supp_i[\theta]$ is strategically equivalent to all elements of $h[\mu, \prod_{\mathcal{B}\in\overline{\mathcal{B}}} supp_{\theta_{\mathcal{B}}}]$. Since $\theta^*[h[\mu, \prod_{\mathcal{B}\in\overline{\mathcal{B}}} supp_{\theta_{\mathcal{B}}}]] = \theta[\mu]$, it follows that the processes are equivalent. □

We now focus on processes in outcome form in which i has no power and with some subprocess decomposition corresponding to a partition $\overline{\mathcal{O}}_i$ of $\mathcal{O}_i$. These processes have all information stripped that are irrelevant for the determination of U_i. Note that the bijection f, every partition of $\mathcal{O}_i$ corresponds to a unique partition of $\mathcal{A}$ and vice versa. The specification of a partition of outcomes gives us a unique subprocess decomposition. In the following, we will refer to such processes alternatively as $\overline{\mathcal{O}}_i$-decomposed or $\overline{\mathcal{A}}$-decomposed processes.

Lemma 8. *Let $\mathcal{S}_i$ contain all processes fulfilling $(G,\theta) = D_{-i}(G,\theta)$ in powerless outcome form with bijection $f : \mathcal{O}_i \to \mathcal{A}$. Let $\overline{\mathcal{O}}_i$ be a partition of $\mathcal{O}_i$ containing three elements with jointly essential outcomes. Then $\overline{\mathcal{A}} = \{f[x] : x \in \mathcal{O}_i\}$ there exists a representation $U : \mathcal{S}_i \to \mathbb{R}$ of $\succsim$ such that:*

$$U[G,\theta] = K\left[\sum_{\mathcal{B}\in\overline{\mathcal{A}}} F_{\mathcal{B}}[U[(G,\theta)|\mathcal{B}], \theta^*_{\overline{\mathcal{A}}}], \theta^*_{\overline{\mathcal{A}}}\right]. \quad (81)$$

Proof. We proceed in the following steps:

1. We provide the representation for an arbitrary fixed partition of outcomes.

2. We show that the choice of the partitioning does not influence the representation.

Consider the decomposition $\overline{\mathcal{A}} = \{\mathcal{B}, \mathcal{C}, \mathcal{D}\}$. Without loss of generality, by the previous lemmas we find the indifferent process $(G^*, \theta^*_{\overline{\mathcal{A}}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\mathcal{D}})$.

We use the result of Gorman, 1968 to obtain an additively separable representation. To apply this result, we require a product space, topological connectedness of each dimension of the product space, and continuous preorders on the subsets of dimensions of the product space that are additively separable in the representation. Clearly, the set of processes of the form $(G^*, \theta^*_{\overline{\mathcal{A}}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\mathcal{D}})$ forms a product space with the dimensions being the strategies over subprocesses and the set of strategies $\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}$ that determine which subprocess is played. We use the preorder topology on these subsets which guarantees connectedness by Lemma 1. By Subprocess Monotonicity, $(G^*, \theta_{\mathcal{B}}) \succsim_i (G^*, \theta'_{\mathcal{B}})$ if and only if $(G^*, \theta_{\overline{\mathcal{A}}} \otimes \theta_{\mathcal{B}} \otimes \theta_{\mathcal{C}} \otimes \theta_{\mathcal{D}}) \succsim_i (G^*, \theta_{\overline{\mathcal{A}}} \otimes \theta'_{\mathcal{B}} \otimes \theta_{\mathcal{C}} \otimes \theta_{\mathcal{D}})$. This yields so-called coordinate independence in each of the subprocesses but not joint independence of the three dimensions. To obtain joint independence, we also need a preorder on combinations of subprocesses, for example $\theta_{\mathcal{B}} \otimes \theta_{\mathcal{C}}$. We obtain this preorder by finding for every process the indifferent process $(G^*, \theta_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}\cup\mathcal{C}} \otimes \theta_{\mathcal{D}})$. For such processes, we have that $(G^*, \theta_{\mathcal{B}\cup\mathcal{C}}) \succsim_i (G^*, \theta_{\mathcal{B}\cup\mathcal{C}})$ if and only if $(G^*, \theta_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}\cup\mathcal{C}} \otimes \theta_{\mathcal{D}}) \succsim_i (G^*, \theta_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}\cup\mathcal{C}} \otimes \theta_{\mathcal{D}})$. This yields a well defined preorder on combinations of subprocesses and therefore we have joint independence. From Gorman, 1968 then follows the existence of a representation of the form:

$$U[G, \theta] = K \left[F_{\mathcal{B}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}, \theta^*_{\mathcal{B}}] + F_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}, \theta^*_{\mathcal{C}}] + F_{\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}, \theta^*_{\mathcal{D}}], \theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}} \right] \tag{82}$$

The extension to arbitrary finite dimensions follows from a simple induction argument, since for a process in powerless outcome form, the union of any two disjoint subgames form a disjoint subgame. Finally, we note that $F_{\mathcal{B}}$ must be an increasing function of $U[(G, \theta)|\mathcal{B}]$ for changes in $\theta^*_{\mathcal{B}}$. □

Lemma 9. *Suppose $(G, \theta) = D_{-i}(G, \theta)$. Let $\overline{\mathcal{A}}$ be a partition of $\mathcal{A}$ into subgames with disjoint, essential outcomes. Then there exists a representation of $\succsim_i$ in the form*

$$U[G, \theta] = \sum_{\mathcal{B} \in \overline{\mathcal{A}}} U[G, \theta^*_{\mathcal{B}}] M_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}}}] + L_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}}}]. \tag{83}$$

The result follows from repeated application of a uniqueness argument of additive representations; if a relation can be represented by a sum of two or more additive functions, then any other such representation must be an affine transformation of this representation.

Proof. By the previous Lemma, we may construct additive representations over processes with decompositions $\overline{\mathcal{A}} = \{\mathcal{B}, \{\mathcal{C}, \mathcal{D}\}, \mathcal{E}\}$ and $\overline{\mathcal{A}}' = \{\mathcal{B}, \mathcal{C}, \{\mathcal{D}, \mathcal{E}\}\}$. Note that on the set of processes in which $\mathcal{E}$ is null, the representations must agree up to a positive monotone transformation. We assume this transformation to be the identity (this is without loss of generality as we can simply redefine K for one of the representations). Letting $\theta^*_{\overline{A}}$ converge to the case where $\mathcal{E}$ is null yields by Continuity two representations of the form:

$$
\begin{aligned}
U[G,\theta] =&K[F_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\overline{\mathcal{A}}}] + F_{\mathcal{C}\cup\mathcal{D}}[U[G,\theta^*_{\mathcal{C}\cup\mathcal{D}}],\theta^*_{\mathcal{B},\mathcal{C}\cup\mathcal{D}}],\theta^*_{\overline{\mathcal{A}}}] \\
=&K'[F'_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\overline{\mathcal{A}}'}] + F'_{\mathcal{C}}[U[G,\theta^*_{\mathcal{C}}],\theta^*_{\overline{\mathcal{A}}'}] + F'_{\mathcal{D}}[U[G,\theta^*_{\mathcal{D}}],\theta^*_{\overline{\mathcal{A}}'}],\theta^*_{\overline{\mathcal{A}}'}] \quad (84)
\end{aligned}
$$

Since the first argument of K and K' are both additive representations over $\theta^*_{\mathcal{B}}$ and $\theta^*_{\mathcal{C}}$, for fixed $\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}$, the transformation $K^{-1}[K'[\cdot,\theta^*_{\overline{\mathcal{A}}'}],\theta^*_{\overline{\mathcal{A}}}]$ must be affine. We note that all information in $\theta^*_{\overline{\mathcal{A}}}$ is contained in $\theta^*_{\overline{\mathcal{A}}'}$. It follows that

$$
F'_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\overline{\mathcal{A}}'}] = F_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\overline{\mathcal{A}}}]M_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}}'}] + L_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}}'}] \quad (85)
$$

Note that we can choose K, K' such that M only depends on $\theta^*_{\overline{\mathcal{A}}\setminus\mathcal{E}} = \theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}$ since the ranking over $\theta^*_{\mathcal{B}}$ is separable from the ranking over $\theta^*_{\{\mathcal{C},\mathcal{D}\}}$. More generally, for any subgame $\mathcal{B}$ and its complement $\mathcal{A}\setminus\mathcal{B}$, the transformation $F_{\mathcal{B}}$ only depends on the utility of the subprocess $U[G,\theta^*_{\mathcal{B}}]$ and the measure $\theta^*_{\mathcal{B},\mathcal{A}\setminus\mathcal{B}}$. Therefore, we obtain the representation:

$$
K'[F'_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\{\mathcal{B},\mathcal{A}\setminus\mathcal{B}\}}] + F'_{\mathcal{C}}[U[G,\theta^*_{\mathcal{C}}],\theta^*_{\{\mathcal{C},\mathcal{A}\setminus\mathcal{C}\}}] + F'_{\mathcal{D}}[U[G,\theta^*_{\mathcal{D}}],\theta^*_{\{\mathcal{D},\mathcal{A}\setminus\mathcal{D}\}}],\theta^*_{\overline{\mathcal{A}'}}] \quad (86)
$$

What is left to show is that all $F_{\mathcal{B}}$ and K are affine. Note that in the above representation, we can let $\theta^*_{\{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}\}}$ converge to $\theta^*_{\{\mathcal{C}\cup\mathcal{D}\}}$ by letting $\mathcal{B}$ become null. On the subset of such processes, we obtain the representation:

$$
U[G,\theta^*_{\mathcal{C}\cup\mathcal{D}}] = K'\left[F'_{\mathcal{C}}[U[G,\theta^*_{\mathcal{C}}],\theta^*_{\{\mathcal{C},\mathcal{D}\}}] + F'_{\mathcal{D}}[U[G,\theta^*_{\mathcal{D}}],\theta^*_{\{\mathcal{C},\mathcal{D}\}}],\theta^*_{\{\mathcal{C},\mathcal{D}\}}\right] \quad (87)
$$

Noting that Equation (84) is a monotone function of the above, we can substitute and use the uniqueness of additive representation argument to obtain that $F_{\mathcal{C}\cup\mathcal{D}}[K'[r,\theta^*_{\{\mathcal{C},\mathcal{D}\}}],\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}]$ is affine in r. This condition fulfills the functional equation solved in Lemma 2. It follows that there exists some continuous monotone transformation of U that makes both K and all functions $F_{\mathcal{C}}$ affine transformations of their first argument. □

Lemma 10.

$$
\begin{aligned}
M_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}] =&M_{\mathcal{C}\cup\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}]M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{D}\}}] \quad (88)\\
=&M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]. \quad (89)
\end{aligned}
$$

Proof. From the previous lemma, we have the following representations:

$$\begin{aligned}U[G,\theta] =&U[G,\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}\otimes\theta^*_{\mathcal{B}}\otimes\theta^*_{\mathcal{C}}\otimes\theta^*_{\mathcal{D}}]\\ =&U[G,\theta^*_{\mathcal{C}}]M_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}]+\dots\\ =&U[G,\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}\otimes\theta^*_{\mathcal{B}}\otimes\theta^*_{\{\mathcal{C},\mathcal{D}\}}\otimes\theta^*_{\mathcal{C}}\otimes\theta^*_{\mathcal{D}}]\\ =&U[G,\theta^*_{\mathcal{C}\cup\mathcal{D}}]M_{\mathcal{C}\cup\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}]+\dots\\ =&U[G,\theta^*_{\mathcal{C}}]M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{D}\}}]M_{\mathcal{C}\cup\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}]+\dots,\end{aligned} \tag{90}$$

where we have assumed without loss of generality that the game form G is sufficiently rich in actions. Using a small change in $U[G,\theta^*_{\mathcal{C}}]$, the first line of the result follows since the change must be equal in all of the above representations. The second line result can be derived by comparing the above representation to:

$$U[G,\theta] =U[G,\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}\otimes\theta^*_{\mathcal{C}}\otimes\theta^*_{\{\mathcal{B},\mathcal{D}\}}\otimes\theta^*_{\mathcal{B}}\otimes\theta^*_{\mathcal{D}}] \tag{91}$$

$$=M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]U[G,\theta^*_{\mathcal{C}}]+\dots \tag{92}$$

By a small change in $U[G,\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]$, it then follows that $M_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{C}\}}]=M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]$. □

Lemma 11. $M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{D}\}}]=\sum_\mu\theta^*_{\{\mathcal{C},\mathcal{D}\}}[\mu]\mu[\mathcal{C}]$.

Proof. We use the special case of a process with a support of three subprocesses and in which only a single strategy μ' yields with positive probability the subprocess obtained by conditioning on the subgame $\mathcal{C}$. Formally, let θ fulfill for all μ:

$$\theta[\mu]>0,\mu[\mathcal{C}]>0\Rightarrow\mu=\mu' \tag{93}$$

from this follows that we can parametrize the following measures:

$$\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}=f[\theta[\mu'],\mu'[\mathcal{C}]] \tag{94}$$

$$\theta^*_{\{\mathcal{C},\mathcal{B}\}}=g\left[\frac{\theta[\mu'](\mu'[\mathcal{B}\cup\mathcal{C}])}{\sum_\mu\theta[\mu](\mu[\mathcal{B}\cup\mathcal{C}])},\frac{\mu'[\mathcal{C}]}{1-\mu'[\mathcal{B}\cup\mathcal{C}]}\right] \tag{95}$$

$$\theta^*_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}}=h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right] \tag{96}$$

We therefore have:

$$M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}] \tag{97}$$
$$=M_{\mathcal{C}}[f[\theta[\mu'],\mu'[\mathcal{C}]]] \tag{98}$$
$$=M_{\mathcal{C}}[\theta^*_{\{\mathcal{B}\cup\mathcal{C}\}}]M_{\mathcal{B}\cup\mathcal{C}}[\theta^*_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}}] \tag{99}$$
$$=M_{\mathcal{C}}\left[g\left[\frac{\theta[\mu'](\mu'[\mathcal{B}\cup\mathcal{C}])}{\sum_\mu \theta[\mu](\mu[\mathcal{B}\cup\mathcal{C}])},\frac{\mu'[\mathcal{C}]}{\mu'[\mathcal{B}\cup\mathcal{C}]}\right]\right]M_{\mathcal{B}\cup\mathcal{C}}\left[h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right]\right] \tag{100}$$

Note that only $M_{\{\mathcal{C},\mathcal{D}\}}$ and $M_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}$ depend on $\mu'[\mathcal{C}]$. We apply the following substitutions:

$$\sum_\mu \theta[\mu]\mu[\mathcal{B}\cup\mathcal{C}] =\hat{p} \tag{101}$$
$$\frac{\theta[\mu']\mu'[\mathcal{B}\cup\mathcal{C}]}{\hat{p}} =\hat{\theta} \tag{102}$$

and obtain:

$$M_{\mathcal{C}}[f[\theta[\mu'],\mu'[\mathcal{C}]]] \tag{103}$$
$$=M_{\mathcal{C}}\left[g\left[\hat{\theta},\frac{\theta[\mu']\mu'[\mathcal{C}]}{\hat{\theta}\hat{p}}\right]\right]M_{\mathcal{B}\cup\mathcal{C}}\left[h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right]\right] \tag{104}$$

It follows that the composition of $M_{\mathcal{B}\cup\mathcal{C}}$ and h only depends on the values of $\theta[\mu']$, $\mu'[\mathcal{B}\cup\mathcal{C}]$, $\hat{p}$, $\hat{\theta}$, which we write as

$$M_{\mathcal{B}\cup\mathcal{C}}\left[h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right]\right] = k[\theta[\mu'],\mu'[\mathcal{B}\cup\mathcal{C}],\hat{p},\hat{\theta}]. \tag{105}$$

Holding $\theta[\mu']$ and $\hat{\theta}$ constant we obtain a Pexider-like logarithmic equation[15] with the solution,

$$M_{\mathcal{C}}[f[\theta[\mu'],\mu'[\mathcal{C}]]] =\hat{f}[\theta[\mu']]\mu'[\mathcal{C}]^\gamma \tag{106}$$
$$M_{\mathcal{C}}\left[g\left[\hat{\theta},\frac{\theta[\mu']\mu'[\mathcal{C}]}{\hat{\theta}\hat{p}}\right]\right] =\hat{g}[\hat{\theta}]\left(\frac{\theta[\mu']\mu'[\mathcal{C}]}{\hat{\theta}\hat{p}}\right)^\gamma \tag{107}$$
$$k[\theta[\mu'],\mu'[\mathcal{C}],\hat{p},\hat{\theta}] =\hat{k}[\theta[\mu'],\hat{\theta}']\,(\hat{p})^\gamma \tag{108}$$

[15] We obtain exactly the Pexider equation on an interval domain by taking logarithms on both sides, rearranging terms, and exponentiating the variables. Solving this functional equation yields the stated result.

Next, we note that $\hat{k}[\cdot] = 1$ since in the limit if $\hat{p} \to 1$, the functions $M_{\mathcal{C}}[f[\ldots]]$ and $M_{\mathcal{C}}[g[\ldots]]$ converge. It follows that $\hat{f}[\theta[\mu']] = (\theta[\mu'])^{\gamma}$ and $\hat{g}[\hat{\theta}] = (\hat{\theta})^{\gamma}$.

Since we have previously obtained in the proof of Lemma 3 an expected utility representation over outcomes, and outcomes are subgames, it follows directly that $\gamma = 1$. □

A.5 Joint Conditional Additivity on Outcomes and Strategies

We now have two representations, one conditionally additively separable in strategies and the other additively separable across outcomes. The two representations are affine transformations of another as shown in the proof of 4. Without loss of generality, we assume this transformation to be the identity transformation. We then obtain the following lemma.

Lemma 12. *If $\overline{\mathcal{A}}$ is a partition of $\mathcal{A}$ into disjoint subgames, then*

$$U_i[G,\theta] = \sum_{\mu_i} \sum_{\mathcal{B}\in\overline{\mathcal{A}}} \theta_i[\mu_i] \left(\mu_i[\mathcal{B}]U[(G,\theta)|\mathcal{B}] + l[\mu_i[\mathcal{B}],\rho[\mathcal{B}]]\right) \tag{109}$$

Proof. We have two representations from previous Lemmas:

$$\sum_{\mu_i} \theta[\mu_i] v_i[\rho|\mu_i,\rho] = \sum_{\mathcal{B}} \rho(\mathcal{B}) U[(G,\theta)|\mathcal{B}] + L_{\mathcal{B}}[\theta] \tag{110}$$

We assume there are four subprocesses, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$. Without loss of generality, we assume that these subprocesses each yield a different outcome with certainty and thus $\rho[x|\mu] = \mu[\mathcal{B}]$ for some outcome x. We can choose a parametrization such that $\rho|\mu_i = f(\epsilon,\delta)$, $\rho|\mu_i' = f'(\epsilon)$, and $\rho|\mu_i'' = f''(\delta)$. Namely, we choose ϵ to transfer probability from $\mu_i[\mathcal{B}]$ to $\mu_i[\mathcal{C}]$ and from $\mu_i'[\mathcal{C}]$ to $\mu_i'[\mathcal{B}]$ to keep the probabilities of the outcomes unchanged. δ reallocates probability from $\mu_i[\mathcal{D}]$ to $\mu_i[\mathcal{E}]$ and from $\mu_i''[\mathcal{E}]$ to $\mu_i''[\mathcal{D}]$. Moreover, $\theta|\mathcal{B} = t(\epsilon)$ and $\theta|\mathcal{B} = t'(\epsilon)$ as well as $\theta|\mathcal{C} = t''(\delta)$ and $\theta|\mathcal{D} = t'''(\delta)$.

$$\theta[\mu_i]v_i[f(\epsilon,\delta),\rho] \tag{111}$$
$$+\,\theta[\mu_i']v_i'[f'(\epsilon),\rho] \tag{112}$$
$$+\,\theta[\mu_i'']v_i''[f''(\delta),\rho] + \ldots \tag{113}$$
$$= L_{\mathcal{B}}[t(\epsilon)] + L_{\mathcal{C}}[t'(\epsilon)] + L_{\mathcal{D}}[t''(\delta)] + L_{\mathcal{E}}[t'''(\delta)] + \ldots \tag{114}$$

and therefore v_i is additively separable in ϵ and δ. Repeating the above steps for reassignments of probability between $\mu_i[\mathcal{B}]$ and $\mu_i[\mathcal{D}]$ as well as between $\mu_i[\mathcal{C}]$ and $\mu_i[\mathcal{E}]$, it is straightforward to obtain that v_i is indeed additively separable across $\mu_i[\mathcal{B}]$, $\mu_i[\mathcal{C}]$, etc.. Thus, $v_i[\rho|\mu_i,\rho] = \sum_{\mathcal{B}} w_{i,\mathcal{B}}[\rho|\mu_i[\mathcal{B}],\rho]$. We can now derive the functional form of $L_{\mathcal{B}}$.

$$L_{\mathcal{B}}[\theta] + L_{\mathcal{C}}[\theta] + \dots \tag{115}$$

$$= \sum_{\mu} \theta[\mu] \left(w_{i,\mathcal{B}}[\mu[\mathcal{B}],\rho] - \mu[\mathcal{B}] U_i[(G,\theta)|\mathcal{B}] \right) \tag{116}$$

$$+ \sum_{\mu} \theta[\mu] \left(w_{i,\mathcal{C}}[\mu[\mathcal{C}],\rho] - \mu[\mathcal{B}] U_i[(G,\theta)|\mathcal{C}] \right) + \dots \tag{117}$$

For fixed $\rho[\mathcal{B} \cup \mathcal{C}]$ and considering only changes in $\theta_{\mathcal{B},\mathcal{C}}$, $L_{\mathcal{B}}$ does not depend on any of the omitted terms. Moreover, $L_{\mathcal{B}}$ does not depend on $\rho[C]$ since $L_{\mathcal{B}}[\theta] = L_{\mathcal{B}}[\theta_{\mathcal{B},\mathcal{C}\cup\mathcal{D}\cup\mathcal{E}}]$. Therefore, for a suitably chosen function $l_{\mathcal{B}}$, we have that $L_{\mathcal{B}}[\theta] = \sum_{\mu} \theta[\mu] l_{\mathcal{B}}[\mu[\mathcal{B}],\rho[\mathcal{B}]]$. By Continuity, we may impose without loss of generality that $l_{\mathcal{B}}[0,\rho[\mathcal{B}]] = 0$.

What is left to show is that $l_{\mathcal{B}}$ can be chosen to be identical across $\mathcal{B}$. For this, suppose that the support of the subprocess $\mathcal{B}$ contains two outcomes. By Continuity, for a sequence of subprocesses such that the probability of one outcome converges to zero, their utility $U_i[(G,\theta)|\mathcal{B}]$ converges to the utility at which the probability of the outcome is zero. Similarly, $U_i[G,\theta]$ converges to the utility at which the probability of the outcome is zero. But then under an outcome equivalent transformation $f_{\mathcal{B}}[\dots] = f_{\mathcal{B}'}[\dots]$ where $\mathcal{B}'$ is obtained by removing actions that yield the outcome. □

A.6 Determination of the Functional Form of Procedural Preferences

We now define $h[x,y] = l[x,y] + l[1-x,1-y]$.

Lemma 13.

$$h(1,x) = \alpha(x \ln[x] + (1-x)\ln[1-x]) \tag{118}$$

Proof. We employ a process such that $\theta[\mu] + \theta[\mu'] + \theta[\mu''] = 1$. Also, $\mu[\mathcal{B}] = \mu'[\mathcal{C}] = \mu''[\mathcal{D}] = 1$. We use the above lemma twice on different partitions,

$\{\mathcal{B}, \mathcal{C} \cup \mathcal{D}\}$ and $\{\mathcal{C}, \mathcal{B} \cup \mathcal{D}\}$. We therefore obtain the two representations:

$$\theta[\mu]h(1,\theta[\mu]) + (1-\theta[\mu])h(0,\theta[\mu]) \tag{119}$$

$$+ (1-\theta[\mu])\left(\frac{\theta[\mu']}{1-\theta[\mu]}h(1,\frac{\theta[\mu']}{1-\theta[\mu]}) + \frac{\theta[\mu'']}{1-\theta[\mu]}h(0,\frac{\theta[\mu'']}{1-\theta[\mu]})\right) \tag{120}$$

$$=\theta[\mu']h(1,\theta[\mu']) + (1-\theta[\mu'])h(0,\theta[\mu']) \tag{121}$$

$$+ (1-\theta[\mu'])\left(\frac{\theta[\mu]}{1-\theta[\mu']}h(1,\frac{\theta[\mu]}{1-\theta[\mu']}) + \frac{\theta[\mu'']}{1-\theta[\mu']}h(0,\frac{\theta[\mu'']}{1-\theta[\mu']})\right) \tag{122}$$

Where we have cancelled the terms containing $U[G, \theta_{\mathcal{B}}]$, etc.. We may assume without loss of generality that $h(1,x) = h(0,x)$. We then have:

$$h(1,\theta[\mu]) + (1-\theta[\mu])\left(h(1,\frac{\theta[\mu']}{1-\theta[\mu]})\right) \tag{123}$$

$$=h(1,\theta[\mu']) + (1-\theta[\mu'])\left(h(1,\frac{\theta[\mu]}{1-\theta[\mu']})\right) \tag{124}$$

This is the fundamental equation of information (Aczél & Dhombres, 1989; Ebanks et al., 1987). Up to a constant and a linear component in $\theta[\mu]$ (which can be removed by redefining $U[G, \theta_{\mathcal{B}}]$), the solution is:

$$h(1,\theta[\mu]) = \alpha\left(\theta[\mu]\ln[\theta[\mu]] + (1-\theta[\mu])\ln[1-\theta[\mu]]\right) \tag{125}$$

□

Lemma 14.

$$h(x,x) = \beta\left(x\ln[x] + (1-x)\ln[1-x]\right) \tag{126}$$

Proof. We employ a process with $\theta[\mu] = 1$, $\mu[\mathcal{B}] + \mu[\mathcal{C}] + \mu[\mathcal{D}] = 1$. The two partitions $\{\mathcal{B}, \mathcal{C} \cup \mathcal{D}\}$ and $\{\mathcal{C}, \mathcal{B} \cup \mathcal{D}\}$ generate two representations which yield after cancelling terms:

$$h(\mu[\mathcal{B}],\mu[\mathcal{B}]) + (1-\mu[\mathcal{B}])\left(h(\frac{\mu[\mathcal{C}]}{1-\mu[\mathcal{B}]},\frac{\mu[\mathcal{C}]}{1-\mu[\mathcal{B}]})\right) \tag{127}$$

$$=h(\mu[\mathcal{C}],\mu[\mathcal{C}]) + (1-\mu[\mathcal{C}])\left(h(\frac{\mu[\mathcal{B}]}{1-\mu[\mathcal{C}]},\frac{\mu[\mathcal{B}]}{1-\mu[\mathcal{C}]})\right) \tag{128}$$

This is the fundamental equation of information with the solution:

$$h(\mu[\mathcal{B}],\mu[\mathcal{B}]) = \alpha\left(\mu[\mathcal{B}]\ln[\mu[\mathcal{B}]] + (1-\mu[\mathcal{B}])\ln[1-\mu[\mathcal{B}]]\right) \tag{129}$$

□

Lemma 15.

$$h[x,y] = (\beta-\alpha)\left(x\ln[x]+(1-x)\ln[1-x]\right) \tag{130}$$
$$+\alpha\left(y\ln[y]+(1-y)\ln[1-y]\right) \tag{131}$$

Proof. We employ a process of the form $\theta[\mu]+\theta[\mu']=1$ with strategies that fulfill: $\mu[\mathcal{B}\cup\mathcal{D}]=1$ and $\mu'[\mathcal{C}]=1$. We obtain the representations:

$$\theta[\mu]h[\mu[\mathcal{B}],\theta[\mu]\mu[\mathcal{B}]]+(1-\theta[\mu])h[0,\theta[\mu]\mu[\mathcal{B}]]+(1-\theta[\mu]\mu[\mathcal{B}]) \tag{132}$$
$$\cdot\left(\frac{\theta[\mu](1-\mu[\mathcal{B}])}{1-\theta[\mu]\mu[\mathcal{B}]}h[0,\frac{1-\theta[\mu]}{1-\theta[\mu]\mu[\mathcal{B}]}]+\left(\frac{1-\theta[\mu]}{1-\theta[\mu]\mu[\mathcal{B}]}\right)h[1,\frac{1-\theta[\mu]}{1-\theta[\mu]\mu[\mathcal{B}]}]\right) \tag{133}$$
$$=\theta[\mu]h[0,1-\theta[\mu]]+(1-\theta[\mu])h[1,1-\theta[\mu]]+\theta[\mu]h[\mu[\mathcal{B}],\mu[\mathcal{B}]] \tag{134}$$

For better readability, we substitute $x=\mu[\mathcal{B}]$ and $y=\theta[\mu]\mu[\mathcal{B}]$.

$$y/xh[x,y]+(1-y/x)h[0,y]+(1-y)h[1,\frac{1-x}{1-y}\frac{y}{x}] \tag{135}$$
$$=h[0,1-y/x]+(y/x)h[x,x] \tag{136}$$

where we made use of $h[0,x]=h[1,x]$. We next solve for $h[x,y]$:

$$h[x,y] = x/yh[0,1-y/x] \tag{137}$$
$$+h[x,x] \tag{138}$$
$$-(x-y)/yh[0,y] \tag{139}$$
$$-x(1-y)/yh[1,\frac{1-x}{1-y}\frac{y}{x}] \tag{140}$$

From Lemmas 13 and 14 we have the solutions for $h[0,y]$, and $h[x,x]$. Substituting these into the above equation gives us the solution for $h[x,y]$:

$$h[x,y] = (\beta-\alpha)\left(x\ln[x]+(1-x)\ln[1-x]\right) \tag{141}$$
$$+\alpha\left(y\ln[y]+(1-y)\ln[1-y]\right) \tag{142}$$
$$=h[x,x]-h[0,x]+h[0,y] \tag{143}$$

It is straightforward to verify that the solutions for $h[x,x]$, $h[0,y]$, and $h[x,y]$ are compatible with another. □

We conclude the proof as follows. If $\beta = 0$, then the procedural preferences are equal to the binary mutual information. We obtain that $\beta = 0$ by lottery independence. If $\beta \neq \alpha$, then $U[D_{\mathcal{N}}(G, \theta)]$ consists of an expectation and an entropy, violating Lottery Independence. Extending the binary mutual information to multiple outcomes follows from substituting the utility representation of the subprocesses. We have therefore identified that U_i is the sum of expectations across outcomes and mutual information. Since we determine the function h for each player, we may choose separate parameters β and name these d_i.

References

Aczél, J., & Dhombres, J. (1989). *Functional Equations in Several Variables*. Cambridge, UK, Cambridge University Press.

Ahlert, M. (2010). A new approach to procedural freedom in game forms. *European Journal of Political Economy*, *26*(3), 392–402. https://doi.org/10.1016/j.ejpoleco.2009.11.003

Bervoets, S. (2007). Freedom of choice in a social context: Comparing game forms. *Social Choice and Welfare*, *29*(2), 295–315. https://doi.org/10.1007/s00355-006-0205-0

Braham, M. (2006). Measuring Specific Freedom. *Economics and Philosophy*, *22*(3), 317–333. https://doi.org/10.1017/S0266267106001003

Burns, S. (2012). *Daybreak of Freedom: The Montgomery Bus Boycott*. Univ of North Carolina Press.

Chen, T.-Y., & Rommeswinkel, H. (2020). Measuring Consumer Freedom.

Chew, S. H., & Sagi, J. S. (2008). Small worlds: Modeling attitudes toward sources of uncertainty. *Journal of Economic Theory*, *139*(1), 1–24. https://doi.org/10.1016/j.jet.2007.07.004

Dowding, K., & van Hees, M. (2009). Freedom of Choice. In *Handbook of Rational and Social Choice*. Oxford, UK, Oxford University Press.

Ebanks, B. R., Kannappan, P., & Ng, C. T. (1987). Generalized fundamental equation of information of multiplicative type. *Aequationes Mathematicae*, *32*(1), 19–31. https://doi.org/10.1007/BF02311295

Erzurumluoglu, A. M., Liu, M., Jackson, V. E., Barnes, D. R., Datta, G., Melbourne, C. A., Young, R., Batini, C., Surendran, P., Jiang, T., Adnan, S. D., Afaq, S., Agrawal, A., Altmaier, E., Antoniou, A. C., Asselbergs, F. W., Baumbach, C., Bierut, L., Bertelsen, S., . . . Howson, J. M. M. (2019).

Meta-analysis of up to 622,409 individuals identifies 40 novel smoking behaviour associated genetic loci. *Molecular Psychiatry*. https://doi.org/10.1038/s41380-018-0313-0

Gorman, W. M. (1968). The Structure of Utility Functions. *The Review of Economic Studies*, *35*(4)jstor 2296766, 367–390. https://doi.org/10.2307/2296766

Gustafsson, J. E. (2010). Freedom of choice and expected compromise. *Social Choice and Welfare*, *35*(1), 65–79. https://doi.org/10.1007/s00355-009-0430-4

Herstein, I. N., & Milnor, J. (1953). An Axiomatic Approach to Measurable Utility. *Econometrica*, *21*(2), 291–297. https://doi.org/10.2307/1905540

Jones, P., & Sugden, R. (1982). Evaluating choice. *International Review of Law and Economics*, *2*(1), 47–65.

Mailath, G. J., Samuelson, L., & Swinkels, J. M. (1993). Extensive Form Reasoning in Normal Form Games. *Econometrica*, *61*(2)jstor 2951552, 273. https://doi.org/10.2307/2951552

Nehring, K., & Puppe, C. (2009). Diversity. In *Handbook of Rational and Social Choice*. Oxford, UK, Oxford University Press.

Neri, C., & Rommeswinkel, H. (2014). *Freedom, Power, and Interference*.

Pattanaik, P. K. (1994). Rights and freedom in welfare economics. *European Economic Review*, *38*, 731–738.

Pattanaik, P. K., & Xu, Y. (1990). On ranking opportunity sets in terms of freedom of choice. *Recherches Economiques de Louvain*, *56*, 383–390.

Peleg, B. (1997). Effectivity functions, game forms, games, and rights. *Social choice and welfare*, *15*(1), 67–80.

Phibbs, C. F. (2009). *The Montgomery bus boycott: A history and reference guide*. Greenwood.

Rommeswinkel, H., & Wu, H. (2020). Monetary Policy Behind the Veil of Ignorance.

Sher, I. (2018). Evaluating Allocations of Freedom. *The Economic Journal*, *128*(612), F65–F94. https://doi.org/10.1111/ecoj.12455

Sugden, R. (2003). Opportunity as a Space for Individuality: Its Value and the Impossibility of Measuring It. *Ethics*, *113*(4), 783–809. https://doi.org/10.1086/373953

Suppes, P. (1996). The nature and measurement of freedom. *Social Choice and Welfare*, *13*(2), 183–200.

Theoharis, J. (2015). *The rebellious life of Mrs. Rosa parks*. Beacon Press.

Wooding, S., Kim, U.-k., Bamshad, M. J., Larsen, J., Jorde, L. B., & Drayna, D. (2004). Natural Selection and Molecular Evolution in PTC, a Bitter-Taste Receptor Gene. *The American Journal of Human Genetics*, *74*(4), 637–646. https://doi.org/10.1086/383092